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ELEMENTS of GEOMETRY

by

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BOOK I.

Definitions.

1. A Point is the mere designation of place in any plane, and is not considered as having any parts or magnitude.

2. A Line is mere length or extension without breadth, and is easily conceived to be produced by the progressive motion of a point.

3. The Extremes or Bounds of a line are points.

4. A Right Line is that which is evenly and without the least deflection to one side or the other, extended between its extremes.

The distinction of a right line from all others lines is easily apprehended from this illustration. If a thread, as representative of mere extension in length, be loosely extended between two ex=

-tremes, it may be disposed at liberty so as to form a variety of lines; but if it be stretched tight between these two extremes, it then only is disposed in a right line.

Cor. Only one right line can be extended between the same two extremes, and this right line is the least which can be extended between the same two extremes.

5. A Surface is extension in length and breadth, and is bounded by a line or lines

6. A plane Surface, or simply a plane, is that which is evenly extended between the lines bounding it.

This may also be illustrated by conceiving a piece of linen, as a representative of surface, to be disposed by the wind, or any other force acting upon it, into a variety of surfaces; but if stretched tight every way between the lines bounding it, it then only is disposed in a plane, or plane surface.

A plane Surface is therefore that, with which right lines, applied upon it in any direction, do entirely coincide.

7. A plane Angle is the inclination of two lines to each other in the same plane, mutually meet:ing each other, and not situated in the same line.

8. A plane Rectilineal Angle is the inclination of two right lines to each other, meeting each other, but not in the same right line.

9. If a right line insisting upon another, makes the adjacent angles equal between themselves, each of these equal angles is called a right angle, and the right line insisting is said to be at right an:gles, or to be Perpendicular to the other.

10. An acute angle is that which is less than a right angle.

11. An Obtuse angle is that which is greater than a right angle.

12. A Figure is a plane surface, bounded by one or more lines.

13. A Circle is a plane figure, bounded by one line called the Circumference, and which line is in every point at an equal distance from a certain point within it, called the Centre.

14. A right line drawn thro' the centre, and terminated on both sides by the circumference, is called a Diameter of the circle.

15. A Semicircle is either part into which a circle is divided by a diameter.

16. A Segment of a circle is either part into which the circle is divided by a right line not passing thro' the centre.

17. Rectilineal Figures are those which are contained by right lines.

18. Trilineal Figures or Triangles, by three right lines.

19. Quadrilateral, by four right lines.

20. An Equilateral triangle is that which has three equal sides.

21. Isosceles, which has two equal sides.

22. Right-angled, which has one right angle.

23. Obtuse angled, which has an obtuse angle.

24. Of four sided figures, a Square is that whose sides are all equal, and angles each a right one.

25. A Rectangle is that, whose sides are not all equal, but whose angles are all right.

26. A Rhombus is that, whose sides are all equal, but whose angles are not right.

27. All other four-sided figures are called Quadrilaterals or Trapezia.

28. Parallel right lines, or parallels, are such as are in the same plane, and are perpendicular to one and the same right line.

29. The distance of a point from a right line is the

perpendicular drawn to it from the point.

Postulates.

1. A Right line may be drawn from any one point to any other point.

2. A right line already drawn of any extent may be produced on either side in the same continued right line to any length.

3. A circle may be described round any point as a centre, which shall pass thro' any other point, or which shall be at any distance from the centre.

Axioms.

1. Things, which are equal to the same, are equal to one another.

2. A magnitude which is greater or less than one of two equal magnitudes, is greater or less than the other.

3. If equal things be added to equal things, the wholes are equal.

4. If equal things be taken from equal things, the remain:ders are equal.

5. Things which are double, or triple, or any equi-mul:tiple of one and the same thing, are equal to one another.

6. Things which are half or third or any equi-part of one and the same thing, are equal to one another.

7. If equal things be added to or taken away from unequal things, the wholes or remainders will be unequal.

8. Magnitudes which applied the one to the other do entirely coincide, are equal between themselves.

9. The whole is greater than the part.

10. Two right lines meet only in one point, or cannot enclose a space.

11. All right angles are equal between themselves.

12. A right line, situate towards one side of another right line, cannot approach nearer to that other

right line, and in its continued progress recede farther from the right line, and vice versâ.

Prop. I. (13. E. 1.)

A right line meeting another on one side of it, makes therewith two angles, which are either each right, or together equal to two right angles.

Let the right line AB meeting the right line CD, (Fig. 1. 2.) make therewith on one side of it the angles ABC, ABD; I say that these angles shall either be each right, or together equal to two right angles.

For they are equal or unequal. If they be equal, (Fig. 1.) the angles ABC, ABD, are each right. (Def. 9.) But if not, let EB, (2.) be the right line insisting on CD, which makes the angles EBC, EBD, equal between themselves, and therefore each right. The angle ABC is equal to the angles EBC, EBA. Add to each the common angle ABD, and the angles ABC, ABD, will together be equal to the

Fig. 5 angles EBC, ABE, ABD, viz; to the angles EBC, EBD, viz: to two right angles.

Cor. (14. E. I.) If two right lines meet another right line in a point to which they tend from different parts, and make the adjacent angles together equal to two right angles, these two right lines shall be in ~~one~~ one right line.

2, Let the right lines CB, DB, meet the right line AB in the point B, to which they tend from the different parts C, D; and make the adjacent angles CBA, DBA, together equal to two right angles; these right lines CB, DB, shall be in one right line.

For if CB, BD be not in one continued right line, continue CB to F. (Post. 2.) The adjacent angles ABC, ABF, are together equal to two right angles (I.) as also the angles ABC, ABD. Therefore the angles ABC, ABF together are equal to the angles ABC, ABD, together; and consequently, taking away the common angle ABC, the remaining

angle ABF will be equal to the remaining angle ABD; viz; Fig.
the part to the whole, which is absurd. Wherefore no right line can be the continuation of CB, unless BD only.

Prop. 2. (15. E. 1.)

If two right lines cut each other, the vertical or opposite angles shall be equal.

If the right lines AB, CD, cut each other in E; I say, the 3.
vertical angles AED, CEB, as also, AEC, BED, shall be equal.

(1) Because the right line AE meets the right line CD in E, the angles AEC, AED, are together equal to two right angles; and also because the right line CE meets the right line AB in E, the angles AEC, CEB, are together equal to two right angles, (I). Therefore the angles AEC, AED, together are equal to the angles AEC, CEB, together; and consequently, taking away the common angle AEC, the remaining angle AED will be equal to the remaining angle CEB. (Ax. 4)

In the same manner it is shown that the angle AEC is equal to the angle BED.

Fig. Cor. 1. If two right lines cut each other, the angles made thereby at the point of section are together equal to four right angles.

Cor. 2. If any number of lines meet or cut in one point, all the angles made thereby are together equal to four right angles.

Prop. 3. (4. E. 1.)

If two triangles have two sides of the one equal to two sides of the other, each to each; and the angle contained by the two sides of the one be equal to the angle contained by the two sides of the other, the remaining or third sides of each shall also be equal; the remaining angles, taken correspondently, shall be equal; and triangle shall be equal to triangle.

4. Let ABC, DEF, be two triangles, wherein the sides AB, BC, of the one are equal to the sides DE, EF, of the other, each to each; viz; AB to DE, and BC to EF; and the angle ABC is also equal to the angle DEF; then

Fig. 4

I say that the remaining sides AC, DF, the remaining angles, taken correspondently, viz; the angles ACB, DFE, and BAC, EDF, as also the triangles themselves shall be equal.

Conceive the triangle ABC to be applied to the triangle DEF, the point B to the point E, and the right line BA to the right line ED. Then, because BA is equal to ED, the angle ABC to the angle DEF, and BC to EF, the point A will coincide with D, the right line BC with the right line EF, and the point C with the point F (Ax. 8.) Wherefore the points A, C, coinciding with the points D, F, the right lines AC, DF, must coincide, because two different right lines cannot be extended between the same extremes (Ax. 10.); and consequently, the angle BAC coincides with the angle EDF, the angle ACB with the angle DFE, and the triangle with the triangle. Wherefore AC is equal to DF, the angle BAC to the angle EDF, the angle ACB to the DFE, and the triangle equal to the triangle (Ax. 8).

Prop. 4. (5. E. I.)

The angles at the base of an Isosceles triangle are equal between themselves.

Let ABC be an isosceles triangle, having the side AB equal to the side AC; I say, the angle ABC shall be equal to the angle ACB.

Produce BA, CA, to b, c, so that Ab be equal to AB, and Ac to AC, and join bc. Because AB, AC, are equal, AB, AC, Ab, Ac, are all equal between themselves. Then AB, AC, are equal to Ab, Ac, each to each, and the angle BAC is equal to the angle bAc (2); wherefore the angle ACB will be equal to the correspondent angle Acb (3). Again, AC, AB, are equal to Ab, Ac, each to each, and the angle BAC is equal to the angle cAb (2.); therefore the angle ABC is also equal to the correspondent angle Acb (3.). Wherefore the angle ABC is equal to the angle ACB, being each equal to the same angle Acb.

Cor. Hence every equilateral triangle is also equiangular.

Prop. V. (6. E. I.)

If two angles of a triangle be equal, the sides subtend-

ing) them will also be equal, that is, the triangle will be isosceles. Fig.

Let ABC be a triangle, having the angle ACB equal to the angle ABC; I say, the side AB shall be equal to the side AC. 6.

If AB be not equal to AC, one of them must be greater. Let AB be the greater, and therefrom be taken BD equal to the less AC, and join CD. Because BD is equal to AC, the two sides BD, BC, of the triangle BDC are equal to the two sides AC, CB, of the triangle BAC, each to each, and the angle DBC is equal to the angle ACB; therefore the triangle, DBC, is equal to the triangle BAC (3.); the part equal to the whole, which is absurd. Wherefore AB is not unequal to AC, that is, it is equal to it.

Cor. Hence every equiangular triangle is also equi~~angular~~lateral.

P. 6. (8. E. I.)

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the angles of the one shall be equal to the correspondent angles of the other.

Let ABC, ABD, be two triangles, having the three sides AB, 7. 8.

Fig 7. 8. AC, BC, of the one, equal to the three sides AB, AD, BD, of the other, each to each, in the order as expressed. I say, any two cor=respondent angles, as ACB, ADB, shall be equal.

Let the triangle ABD approach the triangle ABC, the points A coinciding, as also AB with AB, but the triangles being situ=ate towards different parts of AB. Because AB is equal to AB the points B will coincide. Either therefore BC, BD, are in one right

7. line, or they are not. If BCD be one right line, the whole ACD is an isosceles triangle, because AC is equal to AD, and therefore the angle ACD is equal to the angle ADC (4); that is, the angle ACB is equal to the angle ADB.

8. But if BC, BD, be not in one right line, join CD. Because AC is e=qual to AD, and BC to BD, the triangles ACD, BCD, are each isosceles, and the angles ACD, ADC, as also the angles BCD, BDC, are equal (4.). Consequently the whole or remaining angles ACB, ADB, are also equal (Ax. 3. 4.).

P. 7.

Right lines which are parallel are in every point equidistant,

and a right line perpendicular to one parallel is perpen-dicular to the other also.

Let AB, CD be two right lines parallel to each other, being each perpendicular to the right line EF, and let GH be any right line perpendicular to either of the parallels, as AB, and meet the other CD in H; I say, that GH is equal to EF.

In AB, and on the parts of E opposite to G take EI equal to EG. Draw IL perpendicular to AB, meeting CD in L. Imagine the figure HFEG to turn upon FE, as an axis, and fall towards the figure LFEI. Because the angles at E, F are right, EG will fall upon EI, FH upon FL, and the point G will coincide with the point I, because EG is equal to EI (8. AX.). But also the angles at G, I being right, the right line GH will coincide with the right line. Wherefore the right lines GH, IL, as also FH, FL, coinciding, the points H, L, must coincide, and GH will be equal to IL.

Now, if GH be not also equal to EF, it must be either greater or less. If it be greater, IL is also greater than EF, and therefore

the right line HL, situate on one side of AB, does in its pro-gress from H to F approach nearer to AB, because EF its per-pendicular distance at F is less than HG its perpendicular distance, while continuing its progress in the same di-rection from F to L, it recedes at L to a greater distance from AB, inasmuch as LI is greater than FE. But this is impossible (Ax. 12.) Therefore HG is not greater than EF, nor for the same reason can it be less. The right line HG is therefore equal to EF.

I say farther, that a perpendicular to one parallel is perpendicular also to the other. The same things remaining, bisect EF in I, and draw IO perpendicular to EF. IO will be parallel to each of the right lines AB, CD (Def. 28.). At any point L in IO, draw HLG perpendicular to IO meeting AB in G, CD in H. By the preceding part of this proposition, LG is equal to IE, LH to IF, and therefore the whole GH equal to the whole EF; as also because IE is equal to IF, will LG be equal to LH. Now imagine the figure FILH to turn upon

the right line IL, as an axis, and fall upon the figure EILG. Then in the same manner as in the preceding part, and for the same reasons, will it be demonstrated that FH coincides with and is equal to EG. Join therefore FG, EH. Because in the triangles FEG, FHG, FE, EG, are equal to GH, HF, each to each, and FG is com:mon, the angle FEG will be equal to the angle GHF. But FEG is a right angle, therefore GHF is also a right angle. In the same manner may it be shown that HGE is also a right angle.

The latter part otherwise.

The same things remaining, and EF being the right line to which the parallels are perpendicular, join EH, GF. In the triangles FEG, HGE, FE is equal to HG, GE is common, and the angle FEG is equal to the angle HGE, because each is right; therefore FG is equal to HE (3.). Again in the triangles EFH, GHF, the side EF being equal to the side GH, the side EH to the side GF, and the side FH common, the angle EFH will be equal to the angle GHF (6.). But the angle EFH is a

Fig. 11 right angle, therefore the angle GHF is also right.

Cor. 1. Hence it appears that right lines parallel to one and the same right line are parallel to each other. (30. E. I.)

Cor. 2. And also that right lines perpendicular to either of two parallels, intercept from the parallels equal portions.

P. 8. (29. E. I.)

A right line falling upon two parallels makes the alternate angles equal; the exterior angle equal to the interior and opposite one on the same side; and two interior or two exterior ones on the same side together equal to two right angles.

12. 13. Let a right line EF fall upon the parallels AB, CD, in G, H. I say first, that two alternate angles, EHC, FGB are equal between themselves.

12. If EF be perpendicular to either parallel as AB, it will be perpendicular to the other also (7.) and therefore the angles EHC, FGB, being each right, are equal.

13. But if EF be not perpendicular to either parallel, draw GI

perpendicular to CD, and HI perpendicular to AB. Then GI is equal to HI (7.), GI to IH (Cor. 2.7.) and GH is common to the triangles GIH, HIG, therefore the angle GHI is equal to the angle HGI (6.), that is, the angle EHC is equal to the angle FGB. Whence it is immediately inferred that the angle EGA is also equal to its alternate angle FHD (2.) and the angle EGB equal to its alternate angle FHC (1.).

I say also that an exterior angle EGB is equal to an interior one EHD, on the same side of EF. — For the angle EGB is equal to the alternate angle FHC, and therefore to the angle EHD, which is vertical to FHC (2.).

And lastly I say, that two interior or two exterior angles on the same side of EF, as AGF, CHE, shall together be equal to two right angles. — For the angle AGF being equal to the alternate angle EHD, add to each the angle CHE, and the angles AGF, CHE, will together be equal to the angles EHD, CHE, viz; to two right angles (1.). In the same manner is it proved, that two exterior angles on the same side of EF, as EGB, FHD, are together equal to two right angles.

Prop. 9. (28. E. 1.)

If a right line, falling upon two other right lines, make the alter=nate angles equal, or an exterior angle equal to an interior & opposite one on the same side, or two exterior or two inte=rior angles on the same side together equal to two right angles; these two right lines shall be parallel.

Let the right line EF falling upon the right lines AB, CD, in G, H, make the alternate angles EHC, FGB equal to each other; I say, that AB is parallel to CD.

Draw GI, HL perpendicular to AB. Then GI, HL are parallel, (28. Def.) therefore the angle IGH is equal to the angle LHG (8.), and the angle HGL is equal to the angle GHI. Wherefore the whole angle IHL is equal to the whole angle IGL, viz; to a right angle. The right lines AB, CD are therefore perpendicular to the same right line HL, and therefore are parallel (28. Def.).

Or if an exterior angle FHD be equal to an interior and op=posite angle on the same side, viz; FGB, I say also, that AB, CD, shall be parallel.

For the angle EHC being also equal to the angle FHD (2),

is equal to the angle FGB, which is alternate to it, the right lines AB, CD, are parallel. Fig. 13.

Or lastly, if two interior or two exterior angles on the same side, as FGB, EHD, be together equal to two right angles, AB shall be parallel to CD.

For the angles EHC, EHD are also equal to two right angles (1.); therefore the angles FGB, EHD are together equal to the angles EHC, EHD, and taking away the common angle EHD, the remaining angle FGB will be equal to the remaining alternate angle EHC. Therefore AB, CD, are parallel.

P. 10. (32. E. 1.).

If a side of a triangle be produced, the exterior angle shall be equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, on of whose sides BC is produced to D; 14.
I say, the exterior angle ACD shall be equal to the two interior and opposite angles BAC, ABC.

Thro' C draw CE parallel to AB. The alternate angles ACE, BAC, are e=
=qual, and the exterior angle ECD is equal to ABC, the interior and op:
:posite angle on the same side (8.); therefore the whole angle ACD
is equal to the two angles BAC, ABC together.

I say also, that the three interior angles BAC, ABC, ACB of
any triangle are together equal to two right angles. ~ The same
things remaining, the two angles BAC, ABC together are equal
to the angle ACD, and therefore, adding the common angle
ACB, the three angles BAC, ABC, ACB together are equal to the two angles
ACD, ACB, viz; to two right angles.(1).

Cor. 1. The outward angle of a triangle is greater than either of
the inward and opposite ones (16. E. 1.).

Cor. 2. If one angle of a triangle be right, the other two will together
be equal to a right angle, and therefore each of them acute, or
less than a right angle.

Cor. 3. If two angles of a triangle be equal to two angles of
another, either separately or together, the remaining angle of
the one will be equal to the remaining angle of the other.

Cor. 4. If one angle of a triangle be equal to one angle of another, Fig.
the two remaining angles of the one shall together be equal to the two remaining angles of the other.

Cor. 5. If the angle included by the equal sides of an isosceles triangle be right; each of the remaining angles, viz; at the base, shall be half a right angle.

Cor. 6. Each of the angles of an equilateral triangle is equal to one third of two right angles, or two thirds of one right angle.

Cor. 7. All the interior angles of any rectilineal figure, together with four right angles are equal to twice as many right angles as the figure has sides.

For any rectilineal figure ABCDE can be divided into as many 15.
triangles as the figure has sides, by drawing right lines from a point F within the figure to each of the angles. But, by this proposition, all the angles of these triangles are equal to twice as many right angles, as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure together with the angles at the point

Fig E, that is, together with four right angles (2. C. 2.). Therefore all the ~~interior~~ angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 8. All the exterior angles of any rectilineal figure are equal to four right angles.

Every interior angle ABC together with its adjacent exterior is e=equal to two right angles; therefore all the interior, together with all the exterior angles of the figure are equal to twice as many right angles as the figure has sides; that is, by the preceding corallary, they are equal to all the interior angles of the figure, together with four right angles. Therefore all the exterior angles are equal to four right angles.

P. 11. (26. E. 1.)

If two triangles have two angles of the one equal to two an=gles of the other, each to each, and one side of the one equal to one side of the other, viz; either the side of each interposed between the equal angles, or a side of each opposite to an equal angle of each; the remaining angle of the one shall be equal to the remaining angle of the other; the remaining sides ta=ken correspondently; as also the triangles shall be equal be=tween themselves.

4\. Let ABC, DEF be two triangles, wherein two angles ABC, ACB of the one are equal to two angles DEF, DFE of the other, viz; ABC to

DEF, and ACB to DFE, and first let the side BC interposed between the equal angles be equal, viz; BC be equal to EF; I say that the remaining an:gle BAC shall be equal to the remaining angle EDF, the remaining sides AB, AC, shall be equal to the remaining sides DE, DF, taken correspondently, and the triangle ABC shall be equal to the tri:angle DEF.

Conceive the triangle ABC to be applied to the triangle DEF, the point B to the point E, and the right line BC to the right line EF. The point C will coincide with the point F, because BC is equal to EF (8. Ax.); and therefore at the same time the right lines BA, ED, and CA, FD, will coincide, because the angle ABC is equal to the angle DEF, and the angle ACB to the angle DFE. (8. Ax.). Wherefore the point A will coincide with the point D, because the same two right lines can meet only in one point. (10. Ax.). Consequently the angles BAC, EDF, the sides AB, DE, and AC, DF, as also the triangles themselves, coinciding, are several:ly equal between themselves.

Secondly, let a side in each opposite to an equal angle in each be equal, viz); AB be equal to DE. Because the angles at B, C are equal to the angles at E, F, the angle at A will be equal to the angle at D (3. C. 10.). Wherefore the angles at A, B, being now equal to the angles at D, E, and the interposed side AB equal to the interposed side DE, this second case is reduced to the

Fig. first, and therefore the same conclusion follows.

P. 12.

If two triangles have two sides of the one equal to two sides of the other, each to each; and of the correspondent angles, if two be equal between themselves, but the other two be either each a right angle, or each greater, or each less, than a right angle; these two other angles, as also the remaining sides and angles, shall be equal between themselves.

7. 8. Let the triangles ABC, ABD have the sides AB, AC equal to the sides AB, AD, each to each; and of the correspondent angles, let ACB be equal to ADB, and the other two ABC, ABD be either each a right angle, or each greater, or each less, than a right angle. I say, that the angles ABC, ABD, the sides BC, BD, as also the angles BAC, BAD, shall be equal between themselves.

Let the equal sides AB be applied to each other, and extremity with extremity, but the triangles be situate towards different parts of AB.

If the angles at B be each right; they are equal. Wherefore the angles ACB, ABC, being equal to the angles ADB, ABD, each to each, and of the correspondent sides AC being equal to AD, or AB to AB, the proposition is in this case the same with the eleventh of this.

But if the angles ABC, ABD, be either each greater, or each

less, than a right angle, BC, BD, are not in one continued right line. Join therefore CD. Because AC is equal to AD, the angle ACD is equal to the an=gle ADC (4.), and the angle ACB is equal to the angle ADB; therefore the whole or remaining angle BCD shall be equal to the whole or remain=ing angle BDC, and consequently BC will be equal to BD(5.). Wherefore the three sides of the triangle ABC being equal to the three sides of the triangle ABD, each to each, the correspondent angles ABC, ABD, and BAC, BAD, will be equal between themselves (6.).

P. 13.

A right line, bisecting the vertical angle of an isosceles triangle, bisects also the base, and meets it at right angles; Or, if from the vertex a right line be drawn bisecting the base, it bisects also the vertical angle and meets the base at right angles; Or, if from the vertex a right line be drawn perpendicular to the base, it will bisect the vertical angle and base.

If from the vertex A of an isosceles triangle be drawn AD bisecting the vertical angle BAC; I say, that the base BC will be bisected in D, and AD will be perpendicular to BC. Because AB is equal to AC, AD common, and the angle BAD e=qual to the angle CAD, the base BD will be equal to the base CD, and the angle ADB equal to the angle ADC(3.). But the an=gles ADC, ADB, being thus equal, and also adjacent, are each right (9. Def.). Therefore AD bisects the base, and is at right angles to it.

Or, if AD bisect the base BC, it will also bisect the vertical angle BAC, and be at right angles to the base. ~ For the three sides of the triangle ABD being then equal to the three sides of the triangle ACD, the correspondent angles BAD, CAD, will be equal, viz; the vertical angle will be bisected; and the correspondent angles ADB, ADC will be equal (6.), viz; each will be a right angle.

Or lastly, if AD be at right angles to BC, the vertical angle BAC and also the base BC shall be bisected by AD. ~ For the angles ABD, ACD being equal, because the triangle ABC is isosceles, and the angles ADB, ADC also equal, because each right, and the correspondent side AD being common, the remaining angles BAD, CAD, and remaining sides BD, CD, are equal (11.). Therefore both the vertical angle and base are bisected by AD.

Cor. 1. If a right line perpendicular to the base of a triangle bisect the vertical angle, the triangle shall be isosceles, viz; the sides opposite to the right angle shall be equal, and the base also shall be bisected.

In the triangle BAC, if AD perpendicular to the base BC, bisect the angle BAC, the side AB shall be equal to the side AC, and BD be equal to CD. For, because the angle BAD is equal to the angle CAD, the angle ADB to the angle ADC, and AD common, the triangles ADB, ADC will be equal, AC be equal to AB, and BD be equal to CD (11.).

Cor. 2. If a right line perpendicular to the base of a triangle, at Fig.
the same time ~~[illegible]~~ bisect the base, the triangle will be isosceles,
and the right line will bisect the vertical angle.

The demonstration is equally obvious.

P. 15. (20. E. 1.)

Any two sides of a triangle are together greater than a third.

Let ABC be a triangle; I say, that any two sides of it, as 18.
AB, AC are together greater than the third side BC.

Produce BA to D, making AD equal to AC, and join CD. Then BD is equal to BA, AC together, and because AC is equal to AD, the angle ADC is equal to the angle ACD (4.). But the angle BCD is greater than the angle ACD, as a whole is greater than its part, therefore the angle BCD is greater than the angle ADC or BDC, and consequently BD is greater than BC (14.). But it has been shown that BD is equal to AB, AC together; therefore AB, AC together are greater than BC.

P. 14. (18 & 19. E. 1.)

The greater side of every triangle subtends the greater angle; And the greater angle shall be subtended by the greater side.

Let ABC be a triangle, wherein the side AC is greater than 17.
the side AB; I say, that the angle ABC shall be greater than the angle ACB.

From AC the greater take AD equal to AB the less, and join BD. The angle ABD is equal to the angle ADB (4.); but the angle ADB, being an external angle of the triangle BDC, is greater than the angle DCB or ACB (1.C.10.), therefore the angle ABD is also greater than the angle ACB, and much more so must the angle ABC be.

I say also Conversely, that if one angle, as ABC, of the triangle ABC be greater than another of its angles ACB, the side AC subtending the angle ABC shall be greater than the side AB subtending the angle ACB.

For AC cannot be equal to AB, because then the angle ABC would be equal to the angle ACB (4.), nor can it be less, as then the angle ABC would be less than the angle ACB, neither of which can be. Therefore AC must be greater than AB.

P. 16. (21. E. 1.)

If from the extremities of any one side of a triangle two right lines be inflected to a point not without the triangle, these inflected right lines shall together be less than the other two sides of the triangle, but contain a greater angle.

Let BC be any one side of a triangle ABC, from whose extremities B, C, are inflected BE, CE, first, to a point E in one of the other sides, as AC; I say, that BE, CE, together shall be less than BA, CA together, but contain an angle BEC greater than BAC.

The two sides BA, AE of the triangle BAE are together greater than the third side BE (15.); to each add EC, and BA, AC will be greater than BE, EC. Also, BEC, being an outward angle of the triangle BAE, will be greater than the inward angle BAE or BAC (1.C.10.). – If Secondly, two right lines BD, CD be inflected to a point D within the triangle, I say farther that BD, CD shall be together less than BA, AC, but contain an angle BDC greater than BAC. – Produce BD meeting AC in E. Then, by the preceding, BE, CE are greater than BD, CD, and BA, AC, are greater than BE, EC; therefore BA, AC are much greater than BD, CD. But also, by the preceding, the angle BDC is greater than the angle BEC, and the angle BEC is greater than the angle BAC; therefore the angle BDC is much greater than the angle BAC.

P. 17. (24 & 25. E. 1.).

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them is greater than the angle contained by the two sides of the other; the base of that which has the greater angle shall be greater than the base of the other. And E Contra.

In the triangles ABC, DEF, let AB, AC, be equal to DE, DF, each to each, but the angle BAC be greater than the angle

EDF; I say, BC shall be greater than EF.

Of the sides DE, DF, let DE be the one which is not greater than the other. At the point D draw DG making with DE the angle EDG equal to the angle ABC, and make DG equal to DF, and join FG, EG.

Because the sides AB, AC are equal to the sides DE, DG, and the angle BAC equal to the angle EDG, the base BC is equal to the base EG (3.); and because DF is equal to DG, the angle DFG is equal to the angle DGF (4.). But the angle EFG is greater than the angle DFG, therefore the angle EFG is greater also than the angle DGF, and consequently much greater than the angle EGF. Wherefore in the triangle EFG, the angle EFG being greater than the angle EGF, the side EG is greater than EF (14.); that is, BC is greater than EF.

And Conversely, if the sides AB, AC of one triangle be equal to the sides DE, DF of the other, and BC be greater than EF, the angle BAC shall be greater than the angle DEF.—

For the angle BAC cannot be equal to the angle DEF, as then the base BC would be equal to the base EF (3.), nor can it be less, because then BC would be less than EF (by this); therefore the angle BEC is greater than the angle DEF.

P. 18. (33. E. 1).

The right lines which join the extremities of two equal and parallel right lines, towards the same parts, are themselves equal and

parallel

Let AB, CD be equal and parallel right lines, joined towards the same parts by the right lines AC, BD; I say, AC, BD shall be also equal and parallel. Fig. 21.

Join BC. In the triangles ABC, DCB, AB, BC are equal to DC, BC each to each, and because of the parallels AB, CD, the alternate angles ABC, DCB are equal (8.), therefore AC is equal to BD, (3.) and the angle ACB equal to the angle DBC. But the angles ACB, DBC are the alternate angles made by BC falling upon them, and therefore, being equal, AC will be parallel to BD (9.).

Def. 30. A parallelogram is a quadrilateral, whose opposite sides are parallel.

Def. 31. If the opposite angles of a parallelogram be joined, the point, in which the right lines joining them cut each other, is called the Centre, and every right line drawn thro' the centre and meeting two opposite sides of the parallelogram, is called a Diameter of the parallelogram.

P. 19. (34. E. 1.)

The opposite sides and angles of a parallelogram are equal to each other; and every diameter falling within the parallelogram bisects the parallelogram; and every diameter is itself bisected in the centre.

Let ABCD be a parallelogram; I say First, that the opposite 22.

Fig. sides AB, DC, and AD, BC, as also any two opposite angles, as ABC, ADC, are equal between themselves.

Join AC. In the triangles ABC, ADC, the side AC is common, and because of the parallels AB, DC, and BC, AD, the angles CAB, ACB are respectively equal to the alternate angles ACD, CAD (8.); therefore AB is equal to DC, BC to AD, and the angle ABC to the angle ADC (11.).

I say secondly that every diameter, falling within the parallelogram, bisects the parallelogram or divides it into two equal parts.

When the diameter is a right line joining two opposite angles of the parallelogram, as AC, the reasoning above equally concludes the triangle ABC to be equal to the triangle ADC, and therefore AC bisects the parallelogram.

23 But AC, BD being drawn cutting each other in E, let FG be a diameter meeting two opposite sides AB, CD in F, G. I say AB, BD, FG are each bisected in E, and also that if FG fall within the parallelogram, it shall bisect the parallelogram.

In the triangles AEB, CED, because AB is equal to CD, and the alternate angles EAB, ECD, as also EBA, EDC are equal (8.), AE will be equal to EC, and BE to ED (11.). — Again, because AE is equal to EC, the angle AEF to the vertical angle

CEG (2.), and the angle EAF to the alternate angle ECG (9.); the side EF is e-equal to the side EG (11.), and the triangle AEF to the triangle CEG. Fig.
To each of these equal triangles add the common figure AEGD, and the figure AFGD will be equal to the triangle ACD. Again to each of the equal triangles AEF, CEG add also the common figure EFBC, and the triangle ABC will be equal to the figure BFGC. But the triangle ACD is equal to the triangle ABC, therefore the figure AFGD is equal to the figure BFGC; viz; the parallelogram is bisected by FG.

P. 20. (35 & 36. E. 1.)

Parallelograms upon the same base or equal bases, and between the same parallels are equal to each other.

Let ABCD, EBCF, be two parallelograms, standing upon the same base BC, and between the same parallels AF, BC; I say, that the parallelogram ABCD is equal to the parallelogram EBCF. 24.

Because of the parallelograms, AD, EF are each equal to BC (19.), and therefore equal to each other, and DE is common; therefore the whole or remainder AE is equal to the whole or remainder DF; also AB is equal to DC (19.), and the angle EAB is equal to the angle FDC (9.). Therefore the triangle EAB is equal to the triangle FDC (3.). From the quadrilateral ABCF take away each of these equal triangles, and the remainders, viz; the

Fig. parallelograms ABCD, EBCF, will be equal to each other.

25. I say farther, that two parallelograms ABCD, EFGH standing upon equal bases BC, FG, and between the same parallels AH, BG, are equal to each other.

Join BE, CH. Because BC is equal to FG, and EH is also equal to FG (19.), BC is equal to EH, and they are also parallel; therefore EBCH is a parallelogram (18.). But the parallelograms ABCD, EBCH are equal to each other, because standing upon the same base, and being between the same parallels. For the same reason are the parallelograms EFGH, EBCH equal to each other, because they stand upon the same base EH, and are between the same parallels. Therefore the parallelogram ABCD is equal to the parallelogram EFGH.

P. 21. (37 & 38. E. 1.)

Triangles upon the same base or equal bases, and between the same parallels, are equal to each other.

26. Let ABC, DEF be two triangles, upon the same base BC, or up-
27. on equal bases BC, EF, and between the same parallels BF, AD; they shall be equal to each other.

Draw BG parallel to AC, FH parallel to DE, meeting AD in G, H. The parallelogram GC is double to the triangle ABC (19.), as for the same reason is the parallelogram FH double to the triangle DEF. But the parallelogram GC is equal to the parallelogram

FH(20.); therefore the triangle ABC is also equal to the triangle DEF.

Cor. (41. E.1.). Hence it appears that if a parallelogram and a triangle stand upon the same base, or equal bases, and are between the same parallels; the parallelogram shall be double to the triangle: and therefore that a parallelogram is equal to a triangle, which is between the same parallels, but upon a base double to that of the parallelogram.

P. 22. (39 & 40. E. 1.)

Equal triangles upon the same base, or equal bases, and be- ~~tween the same parallels~~, towards the same parts, are between the same parallels.

~~Demonst.~~ Let two triangles ABC, DEF, equal between themselves, and situate towards the same parts of their common base BF, stand upon the same base, or equal bases BC, EF; I say, that they are between the same parallels.

Join AD. If AD be parallel to BF, the proposition is granted. If it be not parallel, draw AG parallel to BF, meeting DF in G, and join EG. Then the triangles ABC, GEF, being between the same parallels, BF, AG, and standing upon the same base, or equal ba=-ses BC, EF, are equal to each other (21.). But also the triangles ABC, DEF are equal to each other; therefore the triangles DEF, GEF are also equal to each other, the whole to the part, which is absurd. Wherefore no right line drawn thro' A, unless AD

Fig.

only, can be parallel to BF; viz: AD is parallel to BF.

P. 23. (43. E. 1.).

The complements of the parallelograms, which stand about the diameter of any parallelogram, are equal between them-selves.

29. Let ABCD be a parallelogram, AC a diameter of it joining two opposite angles, EH, GF, having the same AC the diameter of each, and therefore said to stand about the diameter AC; then the parallelograms BK, KD which together with EH, GF make up the whole parallelogram BD, are called Comple-ments. I say that BK is equal to KD.

Because AC is a common diameter to the parallelograms BD, EH, GF, the triangle ABC is equal to the triangle ADC, the triangle AEK to the triangle AHK, and the triangle KGC to the triangle KFC (19.). Therefore the whole ABC being equal to the whole ADC, and the parts AEK, KGC to the parts AHK, KFC; the remainder BK shall be equal to the remainder KD, viz; complement to complement.

P. 24.

If on the two sides about any angle of a triangle perpendi-culars be drawn from the opposite angles, the rectangles under each side and the segment intercepted between the angle and the perpendicular shall be equal to each other.

Let ABC be any triangle and on the two sides AB, BC about one of its angles ABC, be drawn from the opposite angles at C, A, the perpendiculars CD, AE. I say that the rectangle under AB, BD, shall be equal to the rectangle under CB, BE. Fig. 30. 31.

Produce CD, AE, to F, H, making FD equal to AB, and EH to BC, and compleat the rectangles DFGB, EHIB. Because DF is equal to AB, and EH to BC, the rectangle BF will be equal to the rectangle ABD, and the rectangle BH to the rectangle CBE. It is therefore to be proved that the rectangle BF is equal to the rectangle BH.

Join CG, AI. Because the angles ABE, CBD are equal, being either the same, or vertical angles, and the angles DBG, EBI, are also equal, being each right, the whole angle CBG is equal to the whole angle ABI. But also BG is equal to AB, and BC to BI, therefore the triangle CBG is equal to the triangle ABI (3). But the triangle CBG and the rectangle BF are upon the same base BG, and between the same par:allels, BG, CF; as also the triangle ABI, and the rectangle BH are upon the same base BI, and between the same paral:lels BI, AH. Therefore the rectangle BF is double to the tri:angle CBG, and the rectangle BH is double to the triangle ABI (Cor. 21). Consequently the triangles being equal, the rect:angles BF, BH will also be equal.

Cor. If from the right angle of a right-angled triangle a perpendicular be drawn to the hypothenuse, the square of either side about the right angle shall be equal to the rectangle under the hypothenuse and the adjacent segment.

31. In the triangle ABC right angled at C, the perpendicular CD is drawn to the hypothenuse AB. Then AC being itself perpendicular to BC, the right lines CB and BE in the proposition become one and the same, and therefore the rectangle CBE is the square of CB. Therefore the rectangle ABD is equal to the square of CB.

In the same manner it appears that the rectangle BAD is equal to the square of AC.

P. 25.

If from the three angles of any rectilineal triangle right lines be drawn perpendicular to the opposite sides, these three perpendiculars shall meet in one common point.

32. Let ABC be any right-lined triangle, and AE, BF, CD be drawn from the angles A, B, C, perpendicular to the opposite sides BC, AC, AB. I say that AE, BF, CD shall have one common concourse.

Because the angles AEB, AFB are each right, the angles EAB, ABF are each less than a right angle (2.C.10.), and therefore are together less than two right angles.

Wherefore AE, BF must meet (C.9). Let them meet in G. For the

same reason must CD, BF meet; and if not in G, let them meet in H, and join CG, AH. In the triangle BCG are drawn CF, GE, perpendicular to the opposite sides, therefore the rectangle GBF will be equal to the rectangle CBE (24). Again, because in the triangle ACB are drawn AE, CD perpendicular to the sides BC, AB, the rectangle CBE will be equal to the rectangle ABD. Therefore the rectangle ABD will be equal to the rectangle GBF. Lastly in the trian-gle ABH, are drawn HD, AF perpendicular to AB, BH, and there-fore the rectangle HBF is also equal to the rectangle ABD. Where-fore the rectangle HBF is equal to the rectangle GBF, the part to the whole, which is absurd. No right line therefore drawn thro' C can be perpendicular to AB, but that which passes also thro' G.

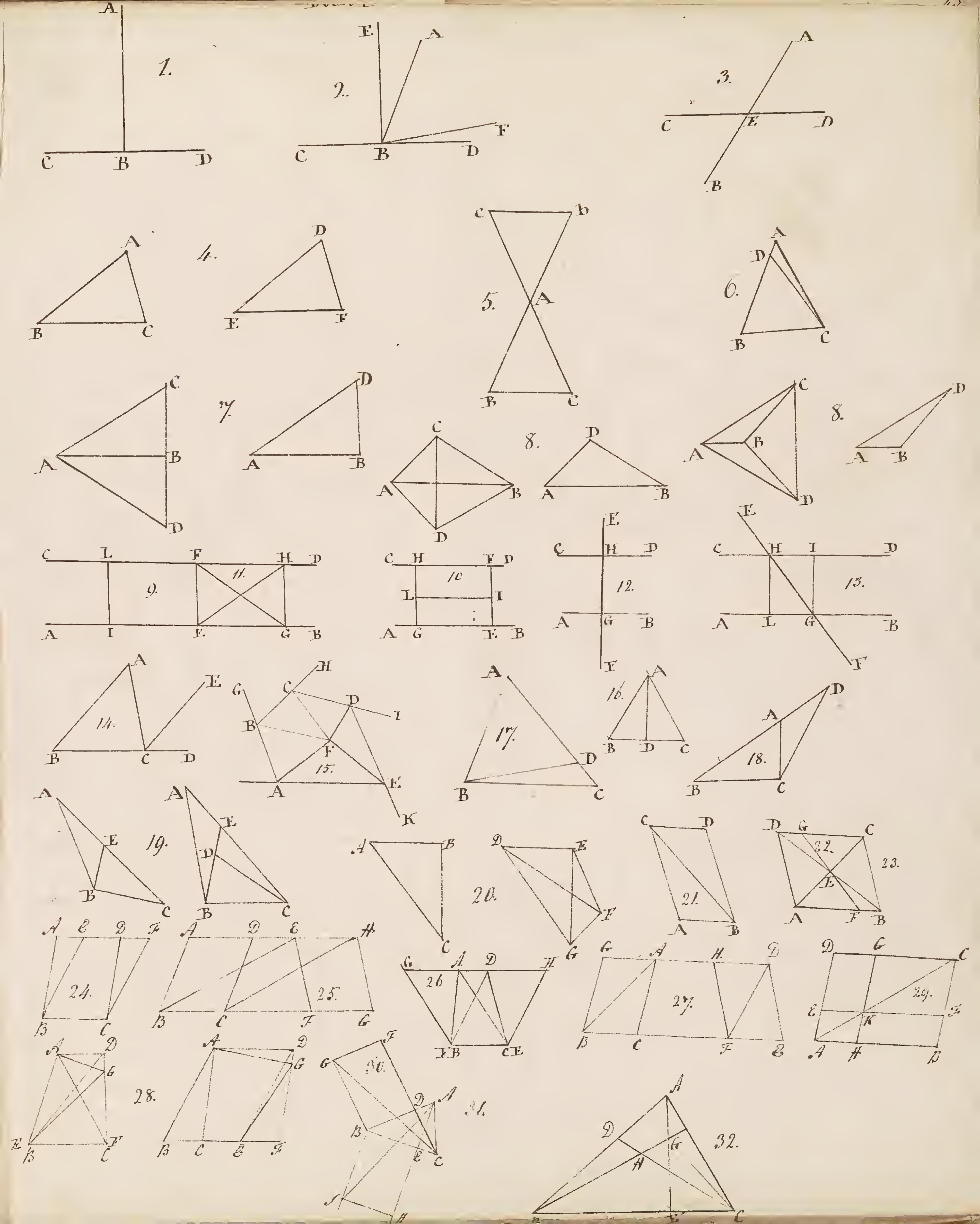
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Fig.

Book II.

Definitions.

1. Every rectangle is said to be contained under any two of its
1. adjoining sides. — Thus the rectangle ABCD is said to be the rectangle under AD, AB, or AD, DC.

Note. When a rectangle under two right lines is signified, and an extremity common to each of them is denoted by the same letter, as the rectangle under AB, AD, it is as a shorter form of expression denoted by the rectangle BAD.

2. In every parallelogram, one of the parallelograms about the diameter, together with the two complements, is called a
2. Gnomon. — Thus the parallelogram HG together with the complements AF, FC, is briefly expressed by the letters AGF, or EHC, which are at the opposite angles of the parallelograms composing the gnomon.

P.1. (1. E. 2.)

If there be two right lines, one of which is divided into any number of parts; the rectangle under the two right lines is equal to all the rectangles under the undivided right line and each part of the divided right line.

3. Let A & BC be two right lines, and BC be divided into parts in D, E; I say, the single rectangle under A, BC is equal to the rectangles under A, BD, under A, DE, and A, EC, together

Draw BG perpendicular to BC, and equal to A; compleat the rect-angle BH; and draw DK, EL parallel to BG or CH. Then BG being perpendicular to BC, and equal to A, DK, EL, HC will also be each perpendicular to BC, and equal to BG, viz; to A (Cor. 8. P. 19. 1.). Consequently BK, DL, EH are each rectangles, and contained under A, BD, under A, DE, and under A, EC, as the whole rectangle BH is that contained under A, BC. But the whole BH is equal to the parts BK, DL, EH, together. Therefore the single rectangle under A, BC is equal to the rectangles under A, BD, under A, DE, and under A, EC, together.

P. 2. (2. E. 2.)

If a right line be divided into two parts, the rectangles under the whole line and each of the parts, shall together be equal to the square of the whole line.

Let AB be a right line divided in C; I say that the rectangles ABC, BAC are together equal to the square of AB.

On AB describe the square ABED, and draw CF parallel to AD or BE. Then because AD, BE are each equal to AB, the rectangle AF is the rectangle under BA, AC, or the rectangle BAC, and the rectangle CE is the rectangle under AB, BC, or the rectangle ABC, as the square AE is the square of AB. But the parts CE, AF are equal to the whole AE; therefore the rectangles ABC, BAC are equal to the square of AB.

Fig.

P. 3. (3. E. 2.)

If a right line be divided into two parts, the rectangle under the whole line and one part shall be equal to the rectangle under the two parts together with the square of the other part.

5. Let the right line AB be divided in C; I say, the rectangle under AB and one part DE is equal to the rectangle under the two parts FD, DE, together with the square of DE.

On BC describe the square CBED, and draw AF parallel to CD or BE, meeting ED produced in F. — Then because CD, BE are each equal to BC, the rectangle AE is that under AB, BC or ABC, and the rectangle AD is that under AC, BC or ACB, as CE is the square of BC. But the whole AE is equal to the parts AD, CE; therefore the rectangle ABC is equal to the rectangle ACB together with the square of BC.

P. 4. (4. E. 2.)

If a right line be divided into two parts, the square of the whole line shall be equal to the squares of the two parts, together with twice the rectangle under the parts.

6. Let AB be a right line divided in C; I say that the square of AB shall be equal to the squares of AC, CB together with twice the rectangle ACB.

Upon AB describe the square ABED, and joining BD,

draw CGF parallel to AD, and HGK parallel to AB. Because BD falls upon the parallels AD, CF, the external angle CGB is equal to the internal and opposite angle ADB (8.1.), and because AB is equal to AD, the angle ABD is also equal to the angle ADB; therefore the angle CGB is equal to the angle CBD or CBG, and consequently CB is equal to CG (2.C.5.1.). But CG being parallel to AD, is perpendicular to AB, therefore the figure CK is a square. For the same reason is HF also a square, and because HG is equal to AC, it is the same as the square of AC, and HF, CK are the squares of AC, CB. Also because CG is equal to CB, the rectangle AG is the rectangle under AC, CB. But the rectangles AG, GE are equal, being complements, (23.1.), therefore the rectangles AG, GE are together double to the rectangle AG, viz; to the rectangle ACB; and as the four figures HF, CK, AG, GE make up the whole figure AE, which is the square of AB; therefore the square of AB is equal to the squares of AC, CB, together with twice the rectangle ACB.

Cor. 1. From the demonstration it appears, that the rectangles about the diameter of a square are likewise squares.

Cor. 2. Hence also it appears, that if a right line be bisected, the square described upon the whole line will be composed of four rectangles, each of which will be squares, each equal to the square described upon half the
~~If a right line be divided into two equal parts, and another~~
line, and therefore that the square described upon the whole line is
~~point be assumed therein, then, First if the point be assumed~~
quadruple to the square described upon half the line.
~~within the limits of the right line, the square of half the~~

P. 5. (7. E. 2.)

line shall be equal to the square shall be equal to the square

Fig.

P. 5. (7. E. 2.)

If a right line be divided into two parts, the square of the whole line together with the square of one of the parts, is equal to twice the rectangle under the whole line and that part, together with the square of the other part.

6. The same things remaining as in the last; I say, that the squares of AB, of either part BC, are equal to twice the rectangle ABC together with the square of AC.

The complement AG is equal to the complement GE, and adding CK to each, AK will be equal to CE, and AK, CE will be equal to twice AK. But AK, CE together are equal to AK, KF, CK together, viz: to the gnomon AKF together with CK. Therefore the gnomon AKF together with CK is equal to twice the rectangle AK, and adding HF, the whole AE together with CK will be equal to twice AK together with HF. But AE is the square of AB, CK is the square of CB (Cor. 4.), AK is the rectangle AB, BC, because BK is equal to BC, and HF is the square of HG, i.e. of AC. Therefore the square of AB together with the square of BC is equal to twice the rectangle ABC together with the square of AC.

P. 6. (5 & 6. E. 2).

If a right line be divided into two equal parts, and another point be assumed therein; then First, if the point be assumed within the terms of the right line, the square of

half the line shall be equal to the square of the intermediate Fig.
part, viz; between the point of bisection and the point assumed,
together with the rectangle under the segments, intercepted
between the terms of the right line, and the point assumed.
But Secondly, if the point be assumed in the right line pro-
-duced, then the square of the intermediate part shall be
equal to the square of half the line together with the rect-
-angle under the segments.

Let the right line AB be bisected in C, and another point 7.
D be assumed therein; either within the terms A, B, or with-
-out in AB produced. On CB describe the square CBFE, and
joining BE, draw DG parallel to CE, meeting EF in G, BE in H.
Also thro' H draw LM parallel to AB meeting CE, BF in M,
L, and AK parallel to CE meeting LM in K. — Because AC
is equal to CB, the rectangle AM is equal to the rectangle
CL (20.1.), and the complement MD is equal to the complement
LG (23.1.). Therefore the whole AH is equal to the gnomon CGL.
In the first case when the point D is between the terms A,
B, add to each of these equal magnitudes AH, and the gno-
-mon CLG, the common rectangle MG, and the rectangles AH,
MG will be equal to the whole CF. But because DL, MG, are
about the diameter of the square CF, the rectangle AH is
the rectangle under AD, DB; MG is the square of MH, viz; of

Fig. CD, and CF is the square of CB. Therefore the rectangle under the segments AD, DB together with the square of the intermediate part CD is equal to the square of CB half the line.

In the second case when the point D is in AB produced, to each of the equal magnitudes AH and the gnomon CLG add the common magnitude CF, and the rectangles AH, CF will together be equal to the whole MG. Therefore as in the first case, the rectangle under AD, DB together with the square of CB half the line, will be equal to the square of CD the intermediate part.

P. 7. (9 & 10. E. 2.).

If a right line be bisected, and another point be assumed either within the terms of the right line, or without in the line produced, the squares of the segments intercepted between the terms of the right line and the point assumed, shall together be double to the squares of half the line, and of the intermediate part.

8 Let the right line AB be bisected in C, and another point D be assumed therein, either within the terms A, B; or without in AB produced. I say that the squares of AD, BD shall together be double to the squares of AC, CD.

Draw CE perpendicular to AB, and equal to AC or CB, join AE, BE, draw DF parallel to CE, meeting BE in F, and FG parallel to AB, meeting CE in G, lastly join AF. ~ Because

the right line BE falls upon the parallels EC, FD, and also BC, FG, the an-gle BFD will be equal to the angle BEC (8.1), and because CB is equal to CE, the angle BEC is equal to the angle EBC or FBD (4.1) therefore the angle BFD is equal to the angle FBD, and BD is equal to FD (5.1). Also on account of the parallels FG, BC, the angle EFG is equal to the angle EBC, and the angle EBC is equal to the angle BEC or FEG; therefore the angle EFG is equal to the angle FEG, and EG is equal to FG, viz; to CD. The square of EG is therefore equal to the square of GF, and the squares of EG, GF are double to the square of GF, or of CD which is equal to it. But the angle EGF being right, the square of EF is equal to the squares of EG, GF (2. Cor. 24.1); therefore the square of EF is double to the square of CD. For the same reason, because CE is equal to AC, and at right angles to it, the square of AE will be double to the square of AC. Produce BE to H. Because the angle EAC is equal to the angle, and EBC equal to the angle BEC, the angles EAB, EBA together are equal to the angle AEB. But the external angle AEH is equal to the angles EAB, EBA together (10.1); there-fore the angle AEH is equal to the angle AEB, and consequent-ly they are each a right angle (2.1). Wherefore the square of AF is equal to the squares of AE, EF together; viz: is double to the squares of AC, CD together. Lastly, because FD is perpen-dicular to AB, the square of AF is equal to the squares

Fig

of AD, DF, that is, because DF is equal to DB, to the squares of AD, DB. Therefore the squares of AD, DB together are double to the squares of AC, CD.

P. 8. (8. E. 2.)

If a right line be divided into two parts, four times the rectangle under the whole line and one of the parts, together with the square of the other part is equal to the square of the right line made up of the whole line and first part.

9. Let a right line AB be divided in C, and either part CB be produced to D, so that BD be equal to BC; I say that four times the rectangle ABC together with the square of AC shall be equal to the square of AD.

On AD describe the square AEFD, and drawing the diameter ED, construct a double figure as in P. 4 & 5.

Then because CO, XH standing about the diameter of a square are themselves squares (Cor. 4. 2.), and CD is bisected in B, the four rectangles CK, BN, GR, KO will each be a square, and equal to each other (2. Cor. 4. 2.); and therefore the whole CO is quadruple of one of them, as CK. Again, because CG is equal to GP, the rectangle AG will be equal to the rectangle MP (20. 1.), as also because PR is equal to RO, the rectangles HR, LO will be equal. But the rectangles MP, HR are equal

Fig.

because they are complements of the rectangle ML (23.1) therefore the four rectangles AG, MP, AR, LO are all equal between themselves, and the four together are quadruple of one of them AG; and consequently the same four together with the four rectangles CK, BN, GR, KO, viz; the gnomon AOH, will be quadruple of the two rectangles AG, CK together, viz; of the one rectangle AK. Add to each the common rectangle XH, and the whole rectangle AF will be equal to four times the rectangle AK, together with the rectangle XH. But the rectangle AF is the square of AD, the rectangle AK is that under AB, BC, because BK is equal to BC, and XH is the square of XP, viz; of AC. Therefore the square of AD is equal to four times the rectangle ABC together with the square of AC.

P. 9. (13. E. 1.)

In every triangle, if on either side containing one of the acute angles a perpendicular be drawn from either of the other angles, the square of the side subtending the acute angle shall be less than the squares of the ~~squares of~~ sides about the acute angle by twice the rectangle under the side on which the perpendicular falls and the segment intercepted between the acute angle and the perpendicular.

Let ABC be any triangle, ACB one of its acute angles, BD a 12.1

perpendicular drawn from either of the other angles on the oppo-site side AC, one of the sides about the acute angle; I say, the square of AB shall be less than the squares of BC, AC by twice the rectangle under AC, CD.

First, let BD fall within the triangle. Draw from the other angles AE, CF perpendicular to BC, AB. The rectangle BAF is equal to the rectangle CAD and the rectangle ABF to the rectangle CBE (24.1), and therefore the square of AB (2.2.) will be equal to the rectangles CAD, CBE. But the rectan-gle CAD is less than the square of AC by the rectangle ACD, and the rectangle CBE is less than the square of BC by the rectangle BCE (2.2); or the rectangle ACD which is equal to it (24.1.) Therefore the rectangles CAD, CBE together are less than the squares of AC, BC together by twice the rectangle ACD, and consequently the square of AB is less than the squares of AC, BC by twice the rectangle ACD.

Secondly, let the perpendicular BD fall without the tri-angle. Then the square of AB together with the rectangle BAF is equal to the rectangle ABF (3.2.). But the rectan-gle BAF is equal to the rectangle CAD, and the rectangle ABF is equal to the rectangle CBE (24.1); therefore the square of AB together with the rectangle CAD is equal to the rectangle CBE. Add to each the common square of AC

and the square of AB together with the rectangle ACD (3.2.) will be equal to the square of AC together with the rectangle CBE. But the rectangle ACD is equal to the rectangle BCE (24.1.), therefore again adding equal things to equal things, the square of AB together with twice the rectangle ACD will be equal to the squares of AC, BC; that is, the square of AB will be less than the squares of AC, BC by twice the rect=angle ACD.

P. 10. (12. E. 2.)

In an obtuse-angled triangle, if from one of the acute angles a perpendicular be drawn to the opposite side, which is one of the sides about the obtuse angle, the square of the side subtending the obtuse angle shall exceed the squares of the sides about the obtuse angle by twice the rectangle under the side on which the perpendicular falls and the segment intercepted between the obtuse angle and the perpendicular.

The same things remaining as in the last, the rectan- 13.
=gle ABF exceeds the square of AB by the rectangle BAF, and the rectangle ACD exceeds the square of AC by the rectangle CAD; therefore the rectangles ABF, ACD together exceed the squares of AB, AC by the rectangles BAF, CAD. But the rectangle ABF is equal to the rectangle

Fig. CBE, the rectangle ACD to the rectangle BCE, and the rectangle BAF to the rectangle CAD (24.1.). Therefore the rectangles CBE, BCE, viz; the square of BC exceeds the squares of AB, AC, by twice the rectangle CAD.

P. 11.

If from any angle of a triangle a right line be drawn bi:secting the opposite side, the squares of the sides about the angle will be double to the squares of the right line drawn, and of half the base.

14. Let ABC be a triangle, from one of whose angles C is drawn CD bisecting the opposite side AB in D; I say, the squares of AC, CB together are double to the squares of CD AD. — Draw CE perpendicular to AB. The square of AC is equal to the squares of CD, AD together with twice the rectangle ADE (10.2.) and the square of CB together with twice the rectangle BDE is equal to the squares of CD, BD (9.2.). But AD is equal to BD, and therefore the squares of AC, CB together with twice the rectangle ADE will be equal to twice the squares of CD, AD together with twice the same rectangle ADE, and taking away this common double rectangle, the squares of AC, CB will be equal to twice the squares of CD, AB.

P. 12.

In every parallelogram, the squares of the four sides are together equal to the squares of the two diagonals.

Let ABCD be a parallelogram whose diagonals AC, BD are drawn; I say, the squares of AB, BC, CD, AD are together equal to the squares of AC, BD.

E being the intersection of AC, BD, the diagonals will be bisected in E (19.1.), and therefore in the triangle ABC, BE being drawn to bisect the base AC in E, the squares of AB, BC are equal to twice the squares of BE, AE (11.). For the same reason are the squares of CD, AD equal to twice the squares of DE, AE, and therefore the squares of AB, BC, CD, AD together equal to four times the square of AE together with twice the squares of BE, DE; that is, because BE is equal to DE, and twice the squares of BE, DE are equal to four times the square of BE; the squares of AB, BC, CD, AD are together equal to four times the squares of AE, BE, viz; e=qual to the squares of AC, BD.

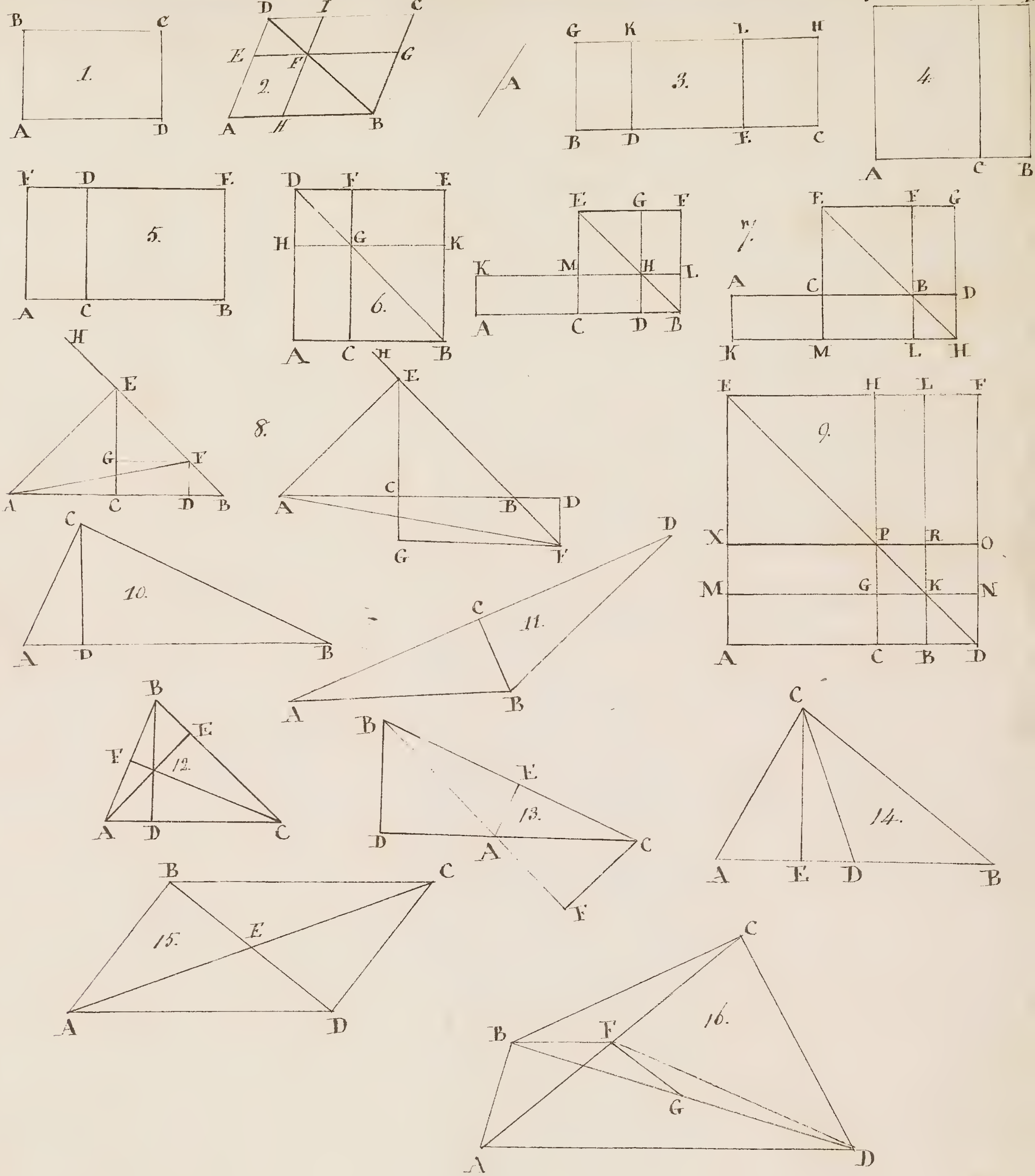
P. 13.

If the diagonals of any quadrilateral be bisected, the squares of the four sides will be equal to the squares of the diagonals together with four times the square of the right line joining the points of bisection.

Let ABCD be a quadrilateral, whose diagonals AC, BD are bisected in F, G. Join FG. I say, the squares of AB, BC, CD, AD are equal to the squares of AC, BD together with four times the square of FG.

Join BF, DF. The squares of AB, BC are equal to twice the squares of AF, BF (11.), as for the same reason are the squares of CD, AD equal to twice the squares of AF, DF; and therefore the squares of AB, BC, CD, AD are equal to four times the square of AF together with twice the squares of BF, DF. Again in the triangle BFD, the squares of BF, DF are equal to twice the squares of BG, FG, and consequently twice the squares of BF, DF are equal to four times the squares of BG, FG. Therefore the squares of AB, BC, CD, AD are equal to four times the squares of AF, BG, FG. But because AC, BD are bisected in F, G, four times the squares of AF, BG are equal to the squares of AC, BD (Cor. 2.) Therefore the squares of AB, BC, CD, AD are equal to the squares of AC, BD together with four times the square of FG.

Scholium. The preceding prop is but a case of this more general one, and is strictly comprehended in the general demonstration of this. For in the 12th the points F, G, coincide, and therefore FG vanishing, the general Prop. becomes the same with the twelfth.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Book III.

Definitions.

1. Equal circles are those whose semi-diameters are equal.
 This is properly a theorem, though so evident, that it approach=es to an axiom; for if two circles, whose semi-diameters are equal be applied to each other, and centre to centre, the circles must coincide, and therefore be equal.
2. A right line is said to Touch a circle, which, meeting it in one point, in every other point falls without the circle.
3. Circles are said to Touch each other which meeting in one point, in every other point fall, either the one within the other or without the other.
4. Right lines are said to be equally distant from the cen=tre of a circle, when the perpendiculars drawn to them from the centre are equal. And that is said to be farther distant from the centre, on which the great=er perpendicular falls.
5. A Segment of a circle is any portion of the circle intercepted by a right line cutting it.
6. The right line intercepting the segment is called the Base or Chord of the segment.
7. An Angle in a Segment is an angle contained by

two right lines drawn from the extremities of the base to any point in the circumference of the segment.

8. The circumference of the other segment is that on which the angle is said to insist or Stand.

9. The Sector of a circle is the figure contained by two semidiameters, and the circumference intercepted by them.

10. Similar segments are those, wherein the angle in the one is equal to the angle in the other.

11. A point is said to be In, Within, or Without, the circumference of a circle, whose distance from the centre is equal to, less, or greater than, the semi diameter of the circle.

Prop. 1. (3. E. 3).

If in a circle a right line drawn thro' the centre meet a right line, not drawn thro' the centre, at right angles, it will bisect it; and if it bisect the right line, it will meet it at right angles.

Let ACB be a circle, AB a right line therein not passing thro' the centre, and EF a right line passing thro' the centre C, and cutting AB at right angles; I say, that AB will be bisected in the point of concourse D. 1.

Join AC, BC, which being semi-diameters of the circle, will be equal to each other, wherefore ACB is an isosceles

Fig. triangle, and therefore EF drawn thro' the vertical angle, and meeting the base AB at right angles in D, will bisect the base in D (10.1.).

And if EF passing thro' the centre, do bisect AB, it will meet it at right angles. For EF being drawn thro' the vertical angle of the isosceles triangle ACB to bisect the base AB, will meet AB at right angles (13.1.).

Cor. 1. A right line in a circle, cutting another right line therein, which is not drawn thro' the centre, at right angles, and also into two equal parts, passes thro' the centre of the circle.

For if not, yet the right line drawn from the centre of the circle to the point of bisection, will also be per:pendicular to the line bisected; that is, two different right lines will be perpendicular to the same right line at the same point, which is absurd.

Cor. 2. If two right lines in a circle mutually bisect each other, the centre of the circle is in the point of bisection.

2.3. Let AB, DE, two right lines in the circle ABE, mutu:ally bisect each other in C, I say, C shall be the cen:tre of the circle.

For if not, let some other point O be conceived to be the centre. Then either, one or neither of the lines

passes thro' C. Let one of them as AB pass thro' O, and therefore be bisected also in O; that is, AB is bisected in two different points, O, C, which is absurd.

If neither of them pass thro' O, join OC. Then OC will be perpendicular to each of the right lines AB, DE (1.3.), which is equally absurd. Therefore no point but C can be the centre of the circle.

Cor. 3. (4.E.3.). If in a circle, two right lines cut each other, and do not both pass thro' the centre, they do not mutually bisect each other.

P. 2. (2. E. 3.)

A right line joining two points in the circumference of a circle, falls within the circle; but if produced on either side, falls without the circle.

Let A, B, be two points in the circumference of a circle ABC; 4. I say, the right line AB joining them falls within the circle.

Let D be any other point in AB, and from C the centre of the circle draw CD, CA, CB. Then the angles ADC, BDC are either each right; or the one obtuse and the other acute. (1.1.). If the angle ADC be either right or obtuse, the side AC subtending it will be greater than the side CD subtending the less angle CAD (14.1.) and therefore the point

D will be within the circle (11. Def. 3). For the same reason will every point in AB unless the points A, B, be within the circle. I say also that AB produced on either side, falls without the circle.

Let E be any point in the right line AB produced, and join CE. Because AC is equal to CB, ACB is an isosceles triangle, and therefore the angle CAB being less than a right angle, the angle CAE will be greater than a right angle (1.1.), and consequently the angle CEA will be less than a right angle, and therefore less than the angle CAE. Therefore CE subtending the angle CAE will be greater than CA subtending the angle CEA (14.1.). The point E is therefore without the circle, as for the same reason is every point in AB produced.

Cor. A right line cannot cut a circle in more than two points.

For if it may cut a circle in more than two points, then would a right line joining two points in the circumference not fall within, or produced, not fall without the circle.

P. 3.

If two circles meet each other in two points, the right line drawn thro' their centres, will cut the

right line joining the two points of concourse into two e=qual parts and also at right angles. Fig.

Let the circles ACB, ADB meet each other in A, B; I say 5.
the right line bisecting AB at right angles will pass thro' the centres of the two circles.

Let AB be bisected in E, and at the point E be drawn CD perpendicular to AB. The centre of each circle is in the right line CD (1. Cor. 1. 3.). Therefore the right line pas=sing thro' the two centres does bisect at right angles the right line joining the two points of concourse; and no right line but one can be drawn thro' the two centres.

Cor. 1. If two circles meet each other in two points, the right line joining their centres will not fall on either point of concourse.

Cor. 2. (10. E. 3.). Two circles cannot meet each other in more than two points.

For, if possible, then the right lines joining their points of concourse being drawn the right lines bisecting them at right angles, would each pass thro' both their centres; that is right lines would meet each other in two points, which is absurd. (10. Ax. 1.)

Cor. 3. (9. E. 3). If from a point within a circle more than two equal right lines be drawn to the circumference, that

Fig.
point is the centre of the circle.

For if not; yet a circle described round that point as a centre, with the distance of any one of these equal right lines, would pass thro' each of the points in which the equal right lines meet the circumference, and if the circle thus described be not one and the same circle with the first, then would two circles meet in more than two points, which is just shewn to be impossible.

P. 4.

If two circles meet in two points, they will cut each other; that is, the circumference of each will fall partly within, partly without the other.

5. Let ABI, ABH be two circles, meeting each other in two points A, B; I say, that they will cut each other in these points; that is, the circumference of each, intercepted between the points of concourse, will fall the one within the other without the other. — Thro' the centres C, D, draw the right line CD, meeting the one circumference in I, G, the other in F, H. CD will not pass thro' either point of concourse A, B (1. Cor. 1. 3.). Draw therefore AC, AD. Then AC, CD are together greater than AD (15. 1.), but AC, CD are together equal to DI, and AD to DF; therefore DI is greater than DF, and so the point I in the circumference AEB is

without the circle ABH. Again AC, AD, or CG, DF, are together greater than CD; therefore taking away CG which is common, DF will be greater than DG, and therefore the point G in the remaining circumference AGB will be within the circle ABH. Wherefore the circle ABI in one part I falls without, and in another part G falls within the circle ABH. By the same reasoning may it be shown that the circle ABH falls partly without and partly within the circle ABI.

Cor. If two circles touch each other, they meet each other only in the point of contact.

For touching each other each falls wholly without or wholly within, the other, but if it were possible that they could meet in two points, each falls partly within, & partly without, the other.

P. 5.

If two circles meet each other, and the right line joining their centres do not fall on the point of concourse, they will meet in another point, and consequently cut each other.

Let the circles AFH, AGI meet each other in A, and the right line CD joining their centres C, D not pass thro' A, I say, the circles will meet in another point, and therefore cut each other. 6.

Draw AE perpendicular to CD, and produce it to B

so that EB be equal to AE. Join CA, CB, and DB, DA. Because AE is equal to BE, EC common, and the angle AEC equal to the angle BEC, being each a right angle, the remaining sides AC, BC are equal between themselves (3.1.). But the point A is in the circumference of the circle AFH; therefore also the point B, being at an equal distance from the centre with A, will be in the circumference AFH. For the same reason is the point B also in the circumference of the circle AGI. Therefore the point B being common to both the circles AFH, AGI, the circles meet again in this other point B, and therefore cut each other.

P. 6. (11 & 12. E. 3.)

If two circles meet each other, and the right line drawn thro' their centres pass thro' the point of concourse, they will touch each other in that point; and if two circles touch each other, the right line drawn thro' their centres will pass thro' the point of contact.

For, if the circles meeting each other do not touch in the point of concourse the one is not wholly internal or wholly external with respect to the other, and therefore they must necessarily meet in some other point. But if circles meet in two points, the right line drawn thro' their centres cannot fall on either of the points of concourse (1. Cor. 1. 3.),

which is absurd, because the right line joining their centres does fall on the point of concourse. Therefore the circles must touch each other in the point of concourse. Fig.

And Conversely, if two circles touch each other, the right line drawn thro' their centres falls on the point of contact. For if not, the circles must meet each other in some other point than the point of contact (4.3.) which is absurd, because touching each other, they can meet only in the point of contact. (Cor. 4.3.). Therefore the ~~right~~ right line drawn thro' their centres must fall on the point of contact.

P. 7. (14. E. 3.)

Equal right lines are equidistant from the centre; and if equidistant, they are equal to each other.

In the circle ABCD whose centre is E, let the right lines AB, CD be equal between themselves. I say that they are equally distant from the centre E.

Draw EF, EG perpendicular to AB, CD. Then AB is bisected in F, and CD in G (1.3.), and therefore because AB is equal to CD, AF will be equal to CG. Join EA, EC. In the triangles AEF, CEG, because the angles at F, G are right, each of the angles AEF, CEG will be less than right (2. Cor. 10.1.); wherefore the sides AE, AF being equal to the sides CE, CG, each to each, and of the correspondent angles the two AFE,

CGE, being equal, and the other two AEF, CEG each less than a right an=
=gle, the remaining sides EF, EG will be equal, (12.1), and there=
=fore AB, CD are equidistant from the centre E.

And if the distances EF, EG be equal between themselves, the right lines AB, CD shall also be equal. For the sides AE, EF being equal to CE, EG, each to each, and of the correspon=
=dent angles, the two AFE, CGE being equal, because each right, and therefore the other two FAE, GCE each less than a right angle, the remaining sides AF, CG, and consequent=
=ly their doubles AB, CD, be equal between themselves.

P. 8. (15. E. 3.)

In a circle the greatest right line is the diameter; and of others, that which is nearer to the centre is greater than that which is more remote; and that which is greater is nearer to the centre than that which is less.

Let ABC be a circle, wherein are inscribed AD, BC, FG, of which AD is a diameter, and BC, FG any two right lines, of which BC is nearer to the centre than FG; I say that AD is greater than BC, and BC greater than FG.

To E the centre of the circle draw BE, CE, and EH, EK perpendicular to BC, FG. Because AE is equal to EB, and ED to EC, the whole AD is equal to EB, EC together. But EB, EC together are greater than BC (15.1), therefore AD is

greater than BC. ~ Again, EB is equal to EF, and the square of EB to the square of EF, that is, the squares of EH, BH together are equal to the squares of EK, KF together (2. Cor. 24. 1.). But because BC is nearer to the centre than FG, EK is greater than EH, and the square of EK greater than the square of EH. Therefore the square of BH, is greater than the square of FK, and BH greater than FK. But BC is double to BH, and FG double to FK (1. 3.); therefore BC is also greater than FG.

And Conversely, if BC be greater than FG, I say BC is nearer to the centre than FG. The same things remaining, because BC is greater than FK, BH will be greater than FK. But the squares of BH, EH together are equal to the squares of FK, EK together, of which the square of BH is greater than the square of FK; therefore the square of EH is less than the square of EK, and EH less than EK. Therefore BC is nearer to the centre than FG.

P. 9. (7. E. 3.)

If from a point within a circle, which is not the centre, right lines be drawn to the circumference, the greatest is that which passes thro' the centre, and the continuation of that right line, compleating the diameter,

is the least; and of others, that which is nearer to the centre is greater than one which is more remote; and from the same point only two right lines can be drawn which are equal to each other, one on each side of the diameter.

Let ABC be a circle, whose centre is E, and F any other point therein, from which are drawn to the circumference FA, FB, FC, FG, FD, of which FA passes thro' the centre E, FB is nearer to FA than FC, and FC than FG, and FD is the continuation of FA, compleating the diameter AD; I say FA is the greatest, FD the least, and FB is greater than FC.

Let FB be any other right line than FA, of those drawn to the circumference from F, and join EB. In the triangle FEB, the two sides EB, EF together are greater than the third side FB (15.1.), but EB is equal to AE, therefore AE, EF together, or AF, is greater than FB. ~ Again join EC, ED. Because EB is equal to EC, EB, EF are equal to EC, EF, but the angle FEB is greater than the angle FEC, therefore the base FB is greater than the base FC (17.1.). For the same reason, is FC greater than FG.

~~Therefore of all the right lines falling upon the concave, FA passing thro' the centre is the greatest, and FB nearer to the centre is greater than FC more remote,~~

Again in the triangle EFG, the side EG is less than the two sides EF, FG; but EG is equal to ED, therefore ED is less than EF, FG together; and taking away EF which is common, the remainder FD will be less than the remainder FG.— Therefore of all the lines, FA is the greatest, FD is the least, and FB is greater than FC, and FC greater than FG.

Lastly, I say that two right lines and only two can be drawn from the point F to the circumference, which shall be equal between themselves, and these will be towards different parts of the diameter AD.

Let FG be any right line drawn from F to the circum=ference. Join EG, and towards opposite parts of AD draw EH, making with the diameter the angle FEH equal to the angle FEG, and join FH. Because EG is equal to EH, the sides EG, EF are equal to the sides EH, EF, each to each, and the angle GEF is equal to the angle HEF; therefore the base FG is equal to the base FH (3.1). And no other right line can be drawn beside FH which shall be equal to FG. For every other right line must be on one side of the diameter, and must be either nearer to or remoter from the diameter than FG or FH; and therefore must be greater or less than FG or FH.

P. 10. (S. E. 3.)

If from a point without a circle right lines be drawn to the circumference, then of those which fall upon the concave circumference, that is the greatest which passes thro' the centre; and of the others that which is nearer to the centre is greater than one which is more remote. But of those which fall upon the convex, the least is that part of the line drawn thro' the centre, which is its ex=:cess above the diameter; and of the others, one which is nearer to the least is less than one which is more remote; And lastly, only two right lines can be drawn from the point to the same circumference, concave or convex, which are equal between themselves, viz; one on each side of the greatest or least lines.

10. Let ABC be a circle, whose centre is E, and F be any point without it, from which are drawn to the concave circumference FA, FB, FC, FG, of which FA passes thro' the centre E; I say, that FA is the greatest, and FB nearer to the centre E is greater than FC more remote, and, for the same reason, that FC is greater than FG.

Join EB, EC, EG. In the triangle FEB, the two sides FE, EB together are greater than the third side FB, (3. 1.), but EB is equal to EA, and FE, EB together

equal to FA, therefore FA is greater than FB. Again, EB is equal to EC, and FE, EB equal to FE, EC, while the angle FEB is greater than the angle FEC; therefore the base FB is greater than the base FC (17.1). For the same reason is FC greater than FG. Therefore of all the right lines falling upon the concave, FA passing thro' the centre is the greatest, and FB nearer to the centre is greater than FC more remote, and FC greater than FG.

Again, if from F be drawn FD, FI, FK, FL, to fall upon the convex, of which FD is the excess of FA above the diameter, and FI is nearer to FD than FK, and FK nearer than FL; I say, that FD is the least, FI less than FK, and FK less than FL.

Join EI, EK, EL. In the triangle FEI, In the triangle FEI, the side FE is less than the two sides FI, EI together (15.1), but the part ED is equal to the part EI, therefore the remainder FD is less than the remainder FI. Again, in the triangle FKE, the right lines FI, IE are drawn from the extremities F, E of the base FE to a point I within the triangle, therefore FI, IE together are less than FK, KE together (16.1). But the part IE is equal to the part KE, therefore the remainder FI is less than the remainder FK. For the same reason, is FK less than FL. ~ Therefore of the

right lines drawn from the point to fall on the convex, FD the excess of FA above the diameter AD is the least, and FI nearer to FD is less than FK more remote, and for the same reason FK is less than FL.

Lastly, I say that if a right line as FI be drawn to the convex, one and only one other equal thereto can be drawn from F to the same convex, and this will be on the parts of FD opposite to FI.

Draw FH, making with FE the angle EFH equal to the angle EFI, let it meet the convex in H, and join EH. Because EI is equal to EH, the sides FE, EI are equal to the sides FE, EH, and the angle FEI is equal to the angle FEH, therefore FI is equal to FH (3.1.). Therefore one right line FH is drawn to the convex equal to FI, and on the other side of FD, and no other can be drawn to the same convex equal to FI or FH, because every other must on one side of FD be nearer or remoter from FD than FI or FH is, and therefore must be less or greater than FI or FH.

P. 11. (16 & 18. E. 3.)

If two right lines meet each other at right angles in the circumference of a circle, and one of them passes thro' the centre, the other shall touch the circle in the point of their concourse; and if two right lines meet

each other in the circumference, of which one passes thro' the Fig.
centre, while the other touches the circle, they shall be at right
angles to each other.

Let ACD be a circle, in whose circumference two right lines CA, 11.
AB meet at right angles viz; in the point A; if CA pass thro' the centre C, the
other AB shall touch the circle in A.

Let B be any other point in AB, and join CB. Because CB subtends the right angle CAB, which angle is greater than the angle ABC (2. Cor. 10.1.), it will be greater than CA (14.1.), and therefore the point B will be without the circle (11. Def. 3.). For the same reason will every point in AB, unless the point A, be without the circle; and therefore AB touches the circle in A. (2. Def. 3.).

And Conversely, if two right lines CA, AB meet each other in A, a point in the circumference, and CA pass thro' the centre, while AB touches the circle in A; I say, that CA will be at right angles to AB.

If not, draw CB perpendicular to AB. Then CBA being a right angle, the angle CAB will be less than a right angle (2. Cor. 10.1.), and therefore CA will be greater than CB, that is, because the point A is in the circumference, the point B will be within the circle (11. Def. 3.). But this is impossible, be-
-cause, AB touching the circle in A, every other point in AB

is without the circle (2. Def. 3.). Therefore CB is not perpendicular to AB, nor for the same reason any right line unless CA.

Cor. 1. (19. E. 3.). If a right line touch a circle and perpendicular thereto at the point of contact another right line be drawn, the centre of the circle shall be in that perpendicular.

For if the perpendicular do not pass thro' the centre, then the right line drawn from the centre to the point of contact would also be perpendicular to the touching line, that is, two right lines would be perpendicular to the same right line at the same point, which is absurd.

Cor. 2. Only one right line can be drawn to touch a circle at the same point.

For if another could be drawn, then would each be perpendicular to the right line drawn from the centre to the point of contact which is absurd as above.

Cor. 3. Every right line meeting a circle, and not touching it in the point of concourse, falls within the circle.

Let AG meet the circle ACD in A, but not touch it in A; I say, that AG falls within the circle. Join CA; which is not perpendicular to AG, because AG does not touch the circle. Draw therefore AF perpendicular to AG. Then, as in the Prop., AC is greater than AF, and therefore the point A being in the circumference, the point F

will be within the circle (11. Def. 3.). Fig.

P. 12. (31. E. 3.).

The angle in a semi-circle is a right angle; in a greater segment, less; and in a less segment, greater than a right angle.

Let ABD be a circle, C its centre, AB a diameter, ADB an angle 12.
in the semi-circle ADB, BDE an angle in the segment BDAE greater than a semi-circle, and BDF an angle in the segment BDF less than a semi-circle; I say, that ADB is a right angle, BDE less than a right angle, and BDF greater than a right angle.

Join CD, and produce BD to G. Because CD is equal to AC, and also to BC, the angle CDA is equal to the angle CAD, and the angle CDB to the angle CBD (4. 1.); therefore the whole angle ADB is equal to the two angles BAD, ABD together. But the angle ADG is also equal to the angles BAD, ABD together (10. 1.), wherefore the angle ADB is equal to the angle ADG, and consequently each is a right angle (9. Def. 1.).

Again, since the segment BDAE is greater than a semi-circle, the remaining segment BE will be less than a semicircle; viz; than BEA; and consequently the point A will be external to the arch BE, and DA be external to the angle BDE. The angle BDE is therefore but a part of the angle BDA, which is a right angle; viz; the angle BDE is

less than a right angle.

Lastly, in the same manner is it shown that the right line DF is external to the angle BDA, and therefore that the angle BDF, of which the right angle BDA is but a part is greater than a right angle.

P. 13. (20. E. 3.).

An angle at the centre of a circle is double to an angle at the circumference, which stands upon the same base, that is, upon the same arc, or part of the circumference.

13\. 14. Let ABD be a circle, ACB an angle at the centre C, and ADB an angle at the circumference, having the same arc or circumference AB as their base. I say, the angle ACB shall be double to the angle ADB.

First, let the angles have one side of each viz; BC BD, in one and the same right line. Because AC is equal to CD, the angle CAD is equal to the angle ADC (4.1.), and therefore the angles CAD, ADC together are double to the angle ADC or ADB. But the external angle ACB is equal to the angles CAD, ADC together are double to the angle ADC or ADB. But the external angle ACB is equal to the angles CAD, ADC, together (10.1.), therefore the angle ACB is also double to the angle ADB.

14 Secondly, let BC, BD not be in one right line.

Join CD, meeting the circumference in E. The angle ACE is double to Fig.
the angle ADE, and the angle BCE double to the angle BDE (by the first case), therefore the whole or remaining angle ACB is double to the whole or remaining angle ADB.

Cor. Angles in the same segment, or what is the same thing, which stand upon the same circumference as a base, are equal between themselves (21. E. 3.).

For they are each the half of the same angle at the centre. This is obvious, when the segment in which the angles are is greater than a semicircle, or when the common circumference on which they stand is less than a semicircle. And a brief illustration will make it equally obvious, when it is the contrary.

Let the angles ADB AEB stand upon the circumference AGB, 15.
greater than a semi circle. Draw DF thro' the centre C, and join EF. Then the circumferences AF, BF are each less than a semicircle, therefore the angle ADF is equal to the angle AEF and the angle BDF to the angle BEF, and consequently the whole angle ADB is equal to the whole angle AEB.

P. 14. (28. E. 3.).

In equal circles, equal right lines subtend equal circumferences, and cut off equal segments, the greater equal to the greater, and the less equal to the less.

17. Let AB, DE be equal right lines in the equal circles ACB, DFE, whose centres are C, F; I say, that the circumferences and segments AGB, DHE, as also AIB, DLE, shall be equal between themselves.

Join AC, BC, DF, EF. Because the circles are equal, AC, CB are equal to DF, FE, each to each, and the base AB is equal to the base DE, therefore the angle ACB is equal to the angle DFE (6.1.). Apply the circle DFE to the circle ACB, the centre F to the centre C, and the right line FD to the right line CA. Because the circles are equal, the circumferences will coincide; the point D will coincide with the point A, because FD is equal to CA, and the right line FE will coincide with the right line CB, because the angle DFE is equal to the angle ACB, and the point E will coincide with the point B, because FE is equal to CB. But the whole circumferences coinciding, and at the same time the points D, A, and E, B, the parts and segments DHE, AGB, as also the remaining parts and segments DLE, AIB, will coincide, and therefore be equal between themselves.

Cor. 1. (27. E. 3.) In equal circles, angles, whether at the centres or at the circumferences, which stand upon equal right lines, are equal between themselves.

When they are at the centres, this is demonstrated in the Prop., and therefore, when they are at the circumferences, they must also be equal, because they are the halves of the equal angles at the centres (13. 3.).

Cor. 2. In the same circle, circumferences or segments subtended by equal right lines, are equal between themselves.

Because they are equal to the same circumference or segment in an equal circle.

Cor. 3. If a figure inscribed in a circle have all its sides equal, the circumferences and segments cut off by each side, will also be all equal.

This is included in the preceding Cor.

P. 15. (29. E. 3.)

In equal circles ~~circumferences~~ equal circumferences are subtended by equal right lines, and therewith include equal segments.

Let ACB, DHE be equal circumferences of the equal circles 17.
ACB, DFE, whose centres are C, F; I say, that the right lines AB, DE, subtending them, and the segments AGB, DHE, intercepted thereby, are equal between themselves.

Apply the circle DFE to the circle ACB, the centre F to the centre C, and the point D to the point A. Because the circles are equal, and the centres coincide, the whole

circumferences will coincide, and therefore because the circumference DHE is equal to the circumference AGB, and the point D coincides with the point A, the point E will coincide with the point B. The right line DE therefore, coinciding in its extremities with those of the right line AB, will be equal to AB, and the segment DHE equal to the segment AGB.

Cor. 1. (26.E.3.). In equal circles, angles which stand upon equal circumferences, whether at the centres or circumferences, are equal between themselves.

For, if the angles be at the centres, they will coincide; and therefore be equal at the same instant that the circumferences are shewn in the Prop. to coincide; but if the angles at the centres be equal, the angles at the circumferences, which are the halves of those at the centres (13.3.) must also be equal.

Cor. 2. Equal circumferences in the same circle, are subtended by equal right lines, and therewith include equal segments.

For they are equal to one and the same thing in an equal circle

Cor. 3. Hence every figure inscribed in a circle, whose sides subtend equal circumferences, has its sides also, and the

segments intercepted by them, equal between themselves.

P. 16. (32. E. 3.).

If a right line touch a circle, and from the point of contact a right line be drawn to cut the circle, the angles made thereby with the touching line, shall be equal to those in the alternate segments of the circle.

Let the right line CD touch a circle ABE in A, and from the point of contact be drawn AB, dividing the circle into the two segments AEB, AFB; I say, that the angle BAC shall be equal to the angle in the alternate segment AEB, and the angle BAD to the angle in the alternate segment AFB.

To any point E in the segment AEB, and to any point F in AFB, draw AE, EB, and AF, BF; draw also the diameter AG, and join GE, GF. AG is perpendicular to CD (11. 3.), and each of the angles AEG, AFG are right (12. 3.). Therefore the angle GAC is equal to the angle AEG, and the part BAG is equal to the part BEG, because they stand upon the same circumference BG (Cor. 13. 3.), wherefore the remaining angle BAC will be equal to the remaining angle AEB.

Again, because the right angle GAD is equal to the right angle AFG, and the angle BAG is equal to the

angle BFG, which stands upon the same circumference BG, the whole angle BAD will be equal to the whole angle AFB.

P. 17. (22. E. 3.)

The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles, and an external angle (made by producing one of its sides) is equal to the internal and opposite angle of the quadrilateral.

Let ABCD be a quadrilateral inscribed in the circle ABCD, one of whose sides AB is produced to E. I say, that two opposite angles ABC, ADC of the quadrilateral shall together be equal to two right angles, and the external angle CBE shall be equal to the internal and opposite angle ADC of the quadrilateral.

Let FG touch the circle in A, and join AC. Because FG touches the circle in A and AC drawn from the point of contact A cuts it, the angle CAG will be equal to the angle ABC in the alternate segment ABC, and the angle CAF be equal to the angle ADC in the alternate segment ADC (16. 3.). Therefore the angles CAG, CAF are together equal to the angles ABC, ADC together. But the angles CAG, CAF are together equal to two right angles (1. 1.), therefore the angles ABC, ADC are also together equal to two right angles.

Again, because the angles ABC, CBE are together equal

to two right angles, the angles ABC, CBE will together be equal to the angles ABC, ADC, and taking away the common angle ABC, the angle CBE will be equal to the angle ADC.

P. 18.

If two equal angles stand upon the same right line and towards the same parts of the right line; or if two angles which together are equal to two right angles, stand upon the same right line but towards different parts of it; the two angular points, and the two extremities of the right line shall be in the circumference of a circle.

Let the angles ABC, ADC stand upon the same right line AC; and be equal between themselves, if towards the same parts of AC, or together be equal to two right angles, if towards different parts of AC; I say, that the four points A, B, C, D, shall be in the circumference of a circle. 19 20

Let a circle be described thro' the three points A, B, C. The circumference of this circle will meet AD in some other point. For, if not, AD must touch this circle in A, and AC being drawn from this same point A to cut this circle, the angle FAC must be equal to the angle ABC in the alternate segment (16.3.), viz; to the angle ADC, in the first case of this Prop.; that is, the external angle FAC of the triangle ACD is equal to ADC, an external and op= 19.

Fig. -posite angle of the triangle, which is absurd (10.1.). Also,
20 in the second case, when the angles ABC, ADC towards op-posite parts of AC, are together equal to two right angles, for the same reason as before, if AD touch the circle in A, the angle DAC is equal to the angle ABC in the alter-nate segment. But the two angles ABC, ADC are together equal to two right angles, therefore the angles DAC, ADC together are also equal to two right angles, that is two angles of a triangle are together equal to two right an-gles, which is absurd (10.1.). Wherefore in neither case does AD touch the circle, and consequently must meet the circle in another point than A. If it does not then meet the circle again in D, let it meet it in some other point E, and join CE.

In the 1st case, because the angles ABC, AEC are in the same segment ABEC, they are equal between themselves (Cor. 15. 1.). But the angle ABC is equal to the angle ADC, therefore the angle ADC is equal to the angle AEC, viz; an outward angle of a triangle equal to an inward and opposite one, which is absurd (10.1.). Also in the second case, because ABCE is a quadrilateral inscribed in a circle, the opposite angles ABC, AEC are together equal to two right angles (17.3.). But the angles ABC, ADC are also together

equal to two right angles, therefore the angle ADC is equal to the angle AEC, the same absurdity as in the 1 case. ~ Therefore the circle does not pass thro' E, nor for the same reason thro' any other point in AD, unless the point D, that is, the four points A, B, C, D, are in a circle.

P. 21.

If two right lines meet each other, whether within the terms of the right lines, or in a point without, and the rectangle under the segments of the one be equal to the rectangle under the segments of the other, viz; understanding by the segments of each, the portions intercepted between the point of concourse, and the terms of each; the four terms shall be in a circumference. Or, if only one term be assigned in one of the right lines, and the square of the segment between this term and the concourse be equal to the rectangle under the segments of the other, the circle described thro' the three terms shall touch the first right line in its single term.

Let AE, BD be two right lines meeting in E, and B, D be 30.
two terms in the one towards the same parts of E, and A be a single term in AE and the square of AE be equal to the rectangle BED. I say that the circle described thro' the three terms A, B, D, shall touch AE in A. –

Thro' C the centre of the circle draw EC cutting the circle in F, G. The rectangle FEG is equal to the rectangle BED (20.3.), and therefore equal to the square of AE. But FC being equal to AC, the square of FC is equal to the square of AC, and consequently the rectangle FEG together with the square of FC is equal to the square of EA, AC. But because FG is bisected in C, and E is a point in FG produced, the rectangle FEG together with the square of FC is equal to the square of EC (6.2.), therefore the square of EC is equal to the squares of EA, AC, ~~But because FG is bisected in C, and E is a point in FG produced, the rectangle FEG together with the square of FC is equal to the square of EC (6.2.), therefore the square of EC is equal to the squares of EA, AC. But because FG is bisected in EC,~~ and consequently the angle EAC is right. Wherefore as the right lines EA, AC, meet at right angles in the circumference of the circle, and one of them AC passes thro' the centre, the other EA will touch the circle in the concourse A (11.3.). (37. E.3.)

Let AC, BD be two right lines, which meet in a point E, either within or without their terms, and the rectangles AEC, BED be equal between themselves; I say, that the four points A, B, C, D, shall be in the circumference of a circle.

Case 1. Let the right lines meet each other within their terms. Imagine a circle to be described, passing thro' the three points A, B, C. Because the point E is between the terms A, C, the right line BE falls within the circle, and therefore must meet the circle again in some other point; if not in D, in some other point F. The rectangle AEC is therefore equal to the rectangle BEF (20.3.) and it is also equal to the rectangle BED. Wherefore the rectangle BED is equal to the rectangle BEF, viz; a part equal to the whole, which is absurd.

Case 2. Let the right lines AC, BD meet in a point E without their terms.

A circle being described as in the 1st case, which passes thro' A, B, C, EB must either touch this circle in B, or cut it. If it be conceived to touch it, the rectangle AEC will be equal to the square of EB, while it is also equal to the rectangle BED; that is, as before, the part is equal to the whole which is absurd. EB does not therefore touch the circle in B, and consequently must meet it in some other point; if not in D, let it meet it in F. The rectangle BEF will then be equal to the rectangle AEC, viz; to the rectangle BED, that is, a part be equal to the whole. Wherefore universally, the circle passes thro' the four points A, B, C, D.

If in the diameter of a circle two points be assumed, equi-distant from the centre, and therefrom be drawn two right lines to meet any where in the circumference, the squares of the right lines drawn will together be double to the squares of the semidiameter and the excentricity of the points.

21. Let E, D be any two points in the diameter of a circle AFB, and equidistant from the centre, from which are drawn EF, DF to any point F in the circumference; I say, that the squares of DF, EF are double to the squares of any semidiameter, AC, and the excentricity CD.

Join FC, which is drawn from ~~the centre of~~ the vertex F of the triangle DFE to bisect the base DE, and therefore the squares of DF, EF are together double to the square of FC or AC, and CD the excentricity (11. 1.).

22. Cor. 1. When the equidistant points are the extremities of the diameter, i.e. when the excentricity becomes the semidiameter itself, the squares of the lines drawn are equal to the square of the diameter.

For the squares of DF, EF are then equal to four times the square of the semidiameter; viz; to the square of the whole diameter. (Cor. 8. 1.)

Cor. 2. If from any two points in the diameter of a circle,

equidistant from the centre, be drawn right lines, two by two, to meet in the circumference; the squares of any two taken toge-ther will be equal to the squares of any other two taken toge-ther.

P. 20.

If two right lines cutting a circle meet each other, (whether within or without the circle), the rectangles under the seg-ments of the cutting lines, intercepted between the circle and the point of concourse, shall be equal between them-selves.

Let AB, CD be two right lines, cutting a circle in A, B, and 23.24.
C, D, and meeting each other in a point E, whether within 25.26 28.29
or without the circle. I say, that the rectangle AEB shall be equal to the rectangle CED.

First, when they meet in a point E within the circle. 23.24 25.25
If, both of them pass thro' the centre as in Fig. 23., the point is the centre, and therefore, and therefore AE, EB, CE, ED being each equal to each other, the rectangles AEB, CED will be equal between themselves.

Again, if one of them only as AB pass thro' the centre S, it may meet the other CD at right angles as in Fig. 24, or not as in Fig. 25. – If AB meet CD at right angles in E, join SD. Because AB is bisected in S, and unequally di-

=vided in E, the rectangle AEB together with the square of ES will be equal to the square of AS (6.2.) or what is the same thing, to the square of SD. Also, because SE is perpendicular to CD, the squares of ED, ES shall together be equal to the square of SD. Therefore the rectangle AEB together with the square of ES is equal to the square of ED together with the square of ES, and taking away the common square of ES, the rectangle AEB will be equal to the square of ED. But CE is equal to ED (1.3.), and consequently the rectangle CED to the square of ED. Therefore the rectangle AEB is equal to the rectangle CED.

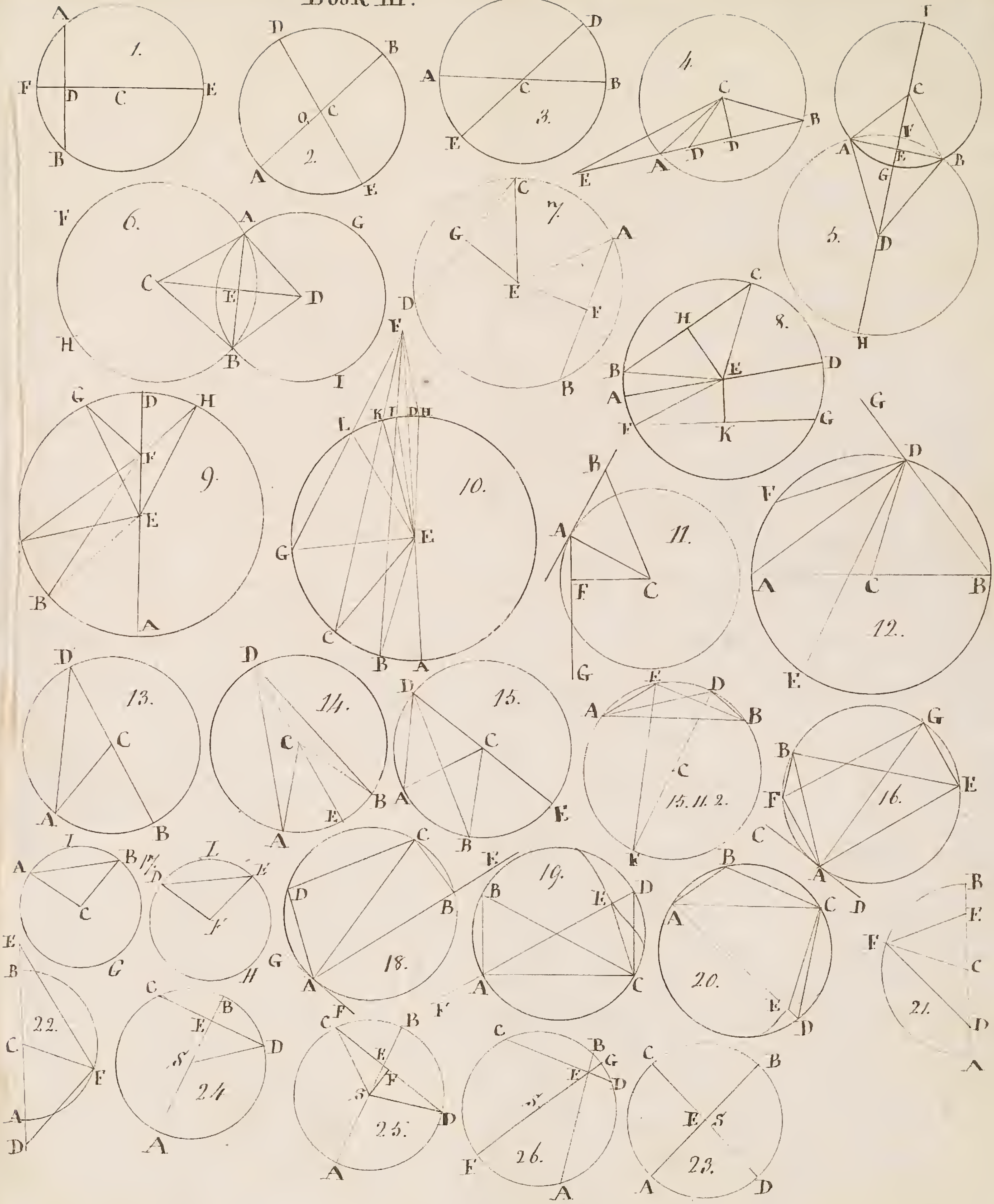
If AB be not perpendicular to CD, draw SF from the centre S perpendicular thereto, and join SD. Then CD will be bisected in F, and being unequally divided in E, the rectangle CED together with the square of EF will be equal to the square of FD. Add to each the common square of SF, and the rectangle CED together with the square of ES will be equal to the square of SD, because the angle at F is right. But as before the rectangle AEB together with the square of ES is equal to the square of AS or of SD. Therefore the rectangle AEB together with the square of ES is equal to the rectangle CED to=gether with the same square of ES, and taking away

this common square, the rectangle AEB will be equal to the rectangle CED.

Now let neither of the right lines AB, CD pass through, 26.
I say that the rectangle AEB shall be equal to the rectangle CED. ~ Thro' the point E draw the diameter FG. Then by the last case, each of the rectangles AEB, CED is equal to the rectangle FEG, and therefore they are equal between themselves

II. When the right lines meet in a point E without the circle. 28.
First, let one of them AB pass thro' the centre S. Draw SF perpendicular to CD, and join SD. CD will be bisected in F, and therefore the rectangle CED together with the square of FD will be equal to the square of FE. Add the common square of SF, and the rectangle CED together with the square of SD will be equal to the square of SE. But because AB is bisected in S, and E is a point in AB produced, the rectangle AEB together with the square of AS is equal to the same square of SE. Therefore the rectangle AEB together with the square of AS is equal to the rectangle CED together with the square of SD. Take away the equal squares of AS, SD and the rectangle AEB will be equal to the rectangle CED.

Wherefore if neither of them pass thro' the centre from the


1. A F D C E B
2. D B O C A E
3. D A C B E
4. C B E A D D
5. I C F A E B G D H
6. F A G C E D H B I
7. C A G E D F B
8. C H B E D A F K G
9. G D H F E B A
10. F L K I D H E G C B A
11. B A F C G
12. G D F A C B E
13. D C A B
14. D C B E A
15. D C A B E
15. 11. 2. E D A B C F
16. G B E F C A D
17. L A B C D E F G H
18. C E D B A F
19. B E D A C F
20. B A C D E D
21. B E F C D A
22. E B C F A D
23. C B E S A D
24. C B E D S A
25. C F B E F S D A
26. C B G E D S F A

concourse E draw the diameter FG. By the preceding case the rectangles AEB, CED are each equal to the rectangle FEG, and therefore equal between themselves.

Cor. 1. If from any point in a circle a right line be drawn at right angles to a diameter of the same, the rectangle under the segments of the diameter shall be equal to the square of the perpendicular.

For in the demonstration of the Prop., ED being perpendicular to the diameter AB, it was demonstrated that the rectangle AEB is equal to the square of ED.

Cor. 2. (36. E. 3.). If a right line cutting a circle meet a right line touching it, the rectangle under the segments of the cutting line shall be equal to the square of the touching line.

27. First, let the right line cutting the circle pass thro' the centre, viz: AB passing thro' the centre S of the circle AGB meet in E the right line EG touching it in G. I say, the rectangle AEB shall be equal to the square of EG. Join SG, which will be perpendicular to EG. Because AB is bisected in S, and E is a point therein produced, the rectangle AEB together with the square of AS will be equal (to the square of SE, viz:) to the squares of EG, SG. Wherefore taking away the equal squares of AS, SG,

Fig.

the rectangle AEB will be equal to the square of EG.

Again, let CD be any other right line cutting the circle in C, D, 28.
and meeting EG in E. The rectangle CED is equal to the rectangle AEB (by this Prop.), and it has just been shown that the rect:angle AEB is equal to the square of EG. Therefore the rectangle CED is also equal to the square of EG.

P. 22.

If two right lines fall upon two concentric circles, the one touching the inner circle, the other cutting them both, the square of either segment of the touching line will be equal to the rectangle under the segments of the cutting line, intercepted between either point of section in either circle, and the circumference of the other.

Let ABD, EGF be two circles, whose common centre is O, and AB, CD be two right lines meeting the outer circle in A, B, & C, D; but AB touching the inner circle in E, while CD cuts it in F, G; I say, that the square of AE or BE shall be equal to the rectangle CFD or FCG.

Join AF, meeting the circles again in I, H, and draw OK perpendicular to AI. Because OK bisects AI, HF in K (1.3.) AK is equal to KI, and HK to KF, and therefore the wholes AF, HI or the remainders AH, FI will be equal between themselves. For the same reason because OE if it were

drawn, would be perpendicular to AB (11.3.), AE will be equal to BE. Therefore the rectangle HAF is equal to the rectangle AFI, and the rectangle CFD equal to the rectangle FCG. But the square of AE is equal to the rectangle HAF, viz; to the rectangle AFI, and the rectangle ~~AFD~~ AFI is equal to the rectangle ~~HCG~~ CFD (20.3). Therefore the square of AE or BE will be equal to the rectangle CFD or FCG.

Cor. 1. If right lines be drawn to fall upon two concentric circles, whether touching or cutting the inner circle, the parts intercepted between a point of concourse in one circle and the circumference of the other will be equal between themselves.

Cor. 2. If right lines cut both circles, the rectangles under the segments of each, intercepted between either point of section in either circle, and the circumference of the other will be equal between themselves.

For they are equal to the square of the same right line.

P. 23.

If a triangle be inscribed in a circle, and from one of its angles two right lines be drawn, making equal angles with the sides, internally or externally, and one of these right lines meet the circumference, while the other meets the base of the triangle, the rectangle

under these lines, so drawn, shall be equal to the rectangle under the sides of the triangle.

Let ACB be a triangle inscribed in the circle ADB, and 32.
from the angle C be drawn CD, CE, making the angle ACD equal to the angle BCE, and the one CD meeting the circle again in D, while the other CE meets the base AB in E; I say, that the rectangle under CD, CE shall be equal to the rectangle ACB.

Produce AC to F, making CF equal to CB, and describe a circle thro' A, D, F, meeting DC in G. Join AD, FG. The angle ADC is equal to the angle CFG, because they stand upon the same circumference (Cor. 13.3.), and the same angle ADC is equal to the angle CBE, because they also stand upon the same circumference AC, or are the one an outward angle, the other an inward and opposite angle of a quadrilateral inscribed in a circle (17.3.). Therefore the angle CFG is equal to the angle ~~B~~ CBE. Again, the angle FCG is equal to the angle ACD (2.1.), and the angle BCE is also equal to the angle ACD, therefore the angle FCG is equal to the angle BCE. In the triangles FCG, BCE therefore two angles of the one are equal to two angles of the other, and the sides CF, CB, interposed between them are also equal, wherefore the side CG is also equal

to the side CE, and the rectangle ACF is equal to the rectangle under AC, CB, and the rectangle DCG equal to the rectangle under DC, CE. But the rectangle DCG is equal to the rectangle ACF (20.3.). Therefore the rectangle under DC, CE, is equal to the rectangle under AC, CB.

P. 24.

If a triangle be inscribed in a circle, and from one of its angles be drawn a right line making equal angles with the sides, internally or externally, and the right line meet the circle and also the base; the rectangle under the segments of the right line intercepted from the vertex of the triangle to the circle and base, shall be equal to the rectangle under the sides of the triangle: but when the right line is drawn external to the triangle, it is requisite that the sides about the angle be not equal between themselves.

33. 34. Let ABC be a triangle inscribed in the circle ABF, and the right line CGE be drawn, making angles ACE, BCG with the sides AC, BC, and these angles equal between themselves. If CD be drawn internally, it falls both within the triangle and the circle, and therefore must meet both the base and the circumference. But if CG be drawn external to the triangle, the sides AC, CB being not equal, let AC be the greater, and therefore the angle ABC greater than the angle BAC (14.1.). Produce AC to H and AB to I. The angle HCG

is equal (to the angle ACE, viz;) to the angle BCG, and consequently the whole angle HCB is double to the angle BCG. But the angle HCB is also equal to the two angles CAB, ABC (10.2.), of which ABC is the greater, and therefore BCG which is the half of HCB, will be less than ABC, and the angles BCG, CBI together will be less than the angles ABC, CBI together, viz; than two right angles (1.1.). Wherefore CG does not meet AB towards those parts where the angles are less (9.1. Cor.). Let it meet it in D. ~ Again, because the angle ABC of the triangle ACB is greater than the angle BAC, the right line CD cannot touch the circle in C, for if it did, then would the angle BAC be equal to the angle BCD (16.3.) CD therefore must meet the circle again in this case, as well as in the preceeding.

In both cases therefore, let CG meet the base AB in D, and the circle in E. Because from the angle C is drawn CD to meet the base in D, and CE to meet the circle in E, and the angle BCD is equal to the angle ACE, the rectangle under CD, CE is equal to the rectangle under AC, CB, the sides of the triangle.

Cor. 1. If a right line bisecting any angle of a triangle meet the base, the rectangle under the segments of the base together with the square of the bisecting line will be equal to the rectangle under the sides of the triangle.

The same things remaining as in the first case, the rectangle ECD is equal to the rectangle under AC, CB, and it is also equal to the rectangle CDE together with the square of CD (3.2.). But the rectangle CDE is equal to the rectangle ADB (20.3.), therefore the rectangle ADB together with the square of CD is equal to the rectangle under AC, CB.

Cor. 2. If from any angle of a triangle the sides about which are not equal, a right line be drawn making equal angles externally with the sides, this right line will meet the base and the rectangle under the segments of the base, intercepted between its terms and the concourse, will be equal to the rectangle under the sides of the triangle together with the square of the right line drawn.

The same things remaining as in the 2nd case, the right line CG making equal angles externally with AC, BC, meets the base in D, and the circle in E. Also the rectangle ECD is equal to the rectangle ACB, and the square of CD is common, therefore the rectangle CDE (3.2.), that is, the rectangle ADB is equal to the rectangle ACB together with the square of CD.

P. 25.

If from any angle of a triangle inscribed in a circle, a perpendicular be drawn to the opposite side, the rectangle under this perpendicular and the diameter shall be equal to the

rectangle under the sides of the triangle about the angle.

Let ABC be a triangle inscribed in a circle, from one of whose an- 30.
=gles C are drawn CD perpendicular to the side AB, and CE the dia- 31.
=meter of the circle. As the angle ABC is less or greater than a right angle, viz; as the segment ABC is greater or less than a semicircle (12.3.), the right lines CD, CE will accordingly fall within or without the triangle. Join AE. Because in the triangles ACE, BCD, the angles CAE, CDB are each right (12.3.), the angles CEA, ACE, of the one are together equal to the angles CBD, BCD of the other (4. Cor. 10.1.). But the angle CEA is equal to the angle CBD, because standing upon the same arch AC (13.3.), or be-cause the one is an inward, the other an outward and op-posite angle of the quadrilateral ABCE inscribed in the circle (17.3.), therefore the remaining angle ACE will be equal to the remaining angle BCD. Wherefore the right lines CD, CE being drawn from the angle C, the one to meet the base AB, the other to meet the circle, and making equal angles, internally or externally, with the sides AC, CB, the rectangle under CD, CE will be equal to the rectangle under AC, CB (23.3.).

Cor. If from any angle of a triangle inscribed in a circle two right lines be drawn, making equal angles with the sides, internally or externally, and the one meet the base, the other meet the circle; also from the same angle a perpendicular be

drawn to the base; the rectangle under the two shall be equal to the rectangle under the perpendicular and the diameter of the circle.

P. 26.

If from the vertex of an isosceles triangle inscribed in a circle, a right line be drawn meeting both the base and the circle, the rectangle under the segments, intercepted from the vertex to the base and circumference shall be equal to the square of either equal side of the triangle.

32.
33. Let ABC be an isosceles triangle inscribed in a circle, having the side AC equal to the side BC, and from the vertex C be drawn CG meeting the base in D and the circle in E; I say the rectangle under CD, CE shall be equal to the square of AC or BC.

Draw CF, internal or external to the triangle ABC, accordingly as CD is drawn, and making with CB the angle BCF equal to the angle ACD. Because AC is equal to BC, the angle CAD to the angle BCF (4.1.), and the angle ACD to the angle BCF, the side CD is equal to the side CF (11.1.), and the rectangle under CD, CE, will be equal to the rectangle under CF, CE. But because CF, CE are drawn the one to meet the base in F, and the other to meet the circle in E, and make equal angles with AC, BC, the rectangle under CF, CE,

will be equal to the rectangle under AC, CB (23.3.) that is to the square of AC or CB. Therefore the rectangle under CD, CE is also equal to the square of AC or BC.

Cor. If an isosceles triangle be inscribed in a circle, and from any point in the base produced two right lines be drawn, the one to the vertex of the triangle, the other to touch the circle, the square of the right line drawn to the vertex shall be equal to the square of either side of the triangle, together with the rectangle under the segments of the base intercepted between the concourse and the terms of the base, or what is the same, together with the square of the touching line.

Every thing remaining the same as in the prop. and D being a point in the base produced, draw DH to touch the circle in H. I say, that the square of CD is equal to the square of AC, together with the rectangle ADB or the square of DH. For the square of DC is equal to the rectan=gles DCE, CDE (2.2.), but the rectangle DCE is equal to the square of AC, and the rectangle CDE is equal to the rectan=gle ADB or to the square of DH (20.3.); therefore the square of DC is equal to the square of AC, together with the rect=angle ADB or the square of DH.

P. 27.

If from any point in the plane of a circle a right line be drawn to cut the circle, and also another perpendicular to a diameter of the same; a relation will exist between the rectangle under the segments of the cutting line, the square of the perpendicular, and the rectangle under the segments of the diameter. Viz;

Case 1. When the point is within the circle.

The rectangle under the segments of the cutting line together with the square of the perpendicular shall be equal to the rectangle under the segments of the diameter.

Case 2. When the point is without the circle, but the perpendicular falls within the terms of the diameter.

The rectangle under the segments of the cutting line together with the rectangle under the segments of the diameter shall be equal to the square of the perpendicular.

Case 3. When the point is also without the circle, but the perpendicular falls without the terms of the diameter.

The rectangle under the segments of the cutting line shall be equal to the rectangle under the segments of the diameter, intercepted from the perpendicular to the terms of the diameter, together with the square of the perpendicular.

34. Case 1.) When the point is within the circle.

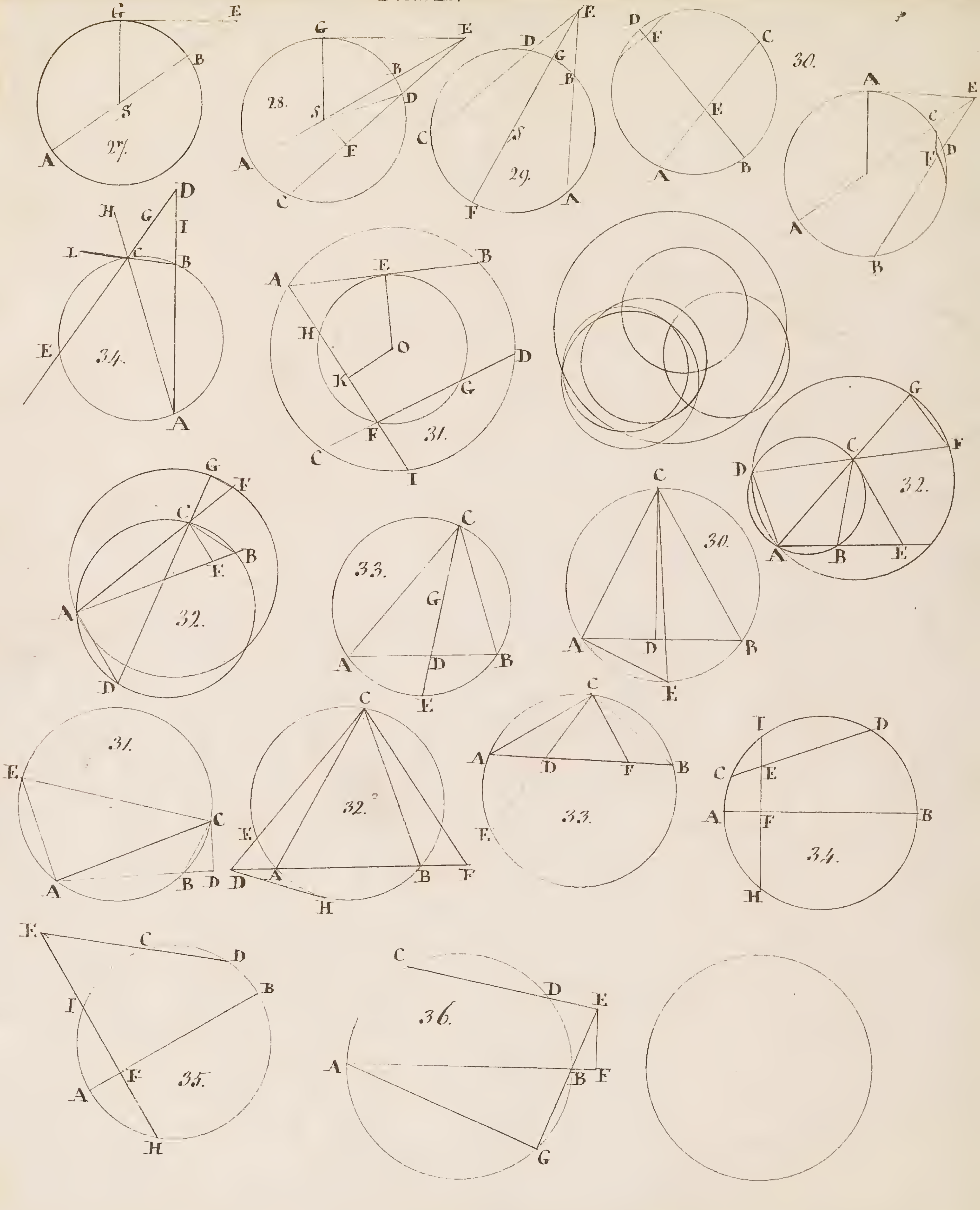
Let ADB be a circle, AB a diameter thereof, and E a point within the circle. From E draw EF perpendicular to AB, and CD cutting the circle in C, D. I say, that the rectangle CED together with the square of EF shall be equal to the rectangle AFB. — Let EF cut the circle in G, H. The rect=:angle CED is equal to the rectangle GEH (20.3; add to each the square of EF, and the rectangle CED together with the square of EF is equal to the rectangle GEH together with the square of EF, viz; because GH is bisected in F to the square of GF (6.2.). But the square of GF is equal to the rectangle AFB (Cor. 20.3.), therefore the rectangle CED together with the square of EF is equal to the rectangle AFB.

Case. 2.) When the point E is without the circle, but EF meets the diameter within the terms A, B. 35.

Every thing also remaining the same, the rectangle CED together with the square of GF is now equal to the rectangle GEH together with the square of GF. But the square of GF is equal to the rectangle AFB, and the rect-angle GEH together with the square of GF, is equal to the square of EF. Therefore the rectangle CED together with the rectangle AFB is equal to the square of EF.

Case 3. When the point E is without the circle, but EF meets the diameter without the terms A, B. 36.

I say, that the rectangle CED is equal to the rectangle AFB together with the square of EF. — Draw EB meeting the circle again in G, and join AG. Because AG is perpendicular to EB (12. 3.), and EF is perpendicular to AB, the rectangle ABF is equal to the rectangle EBG (24. 1.), and the squares of BF, EF are equal to the square of EB (A. 2.), therefore the rectangle ABF together with the squares of BF, EF is equal to the rectangle EBG together with the square of EB. But the rectangle ABF together with the square of BF is equal to the rectangle AFB, and the rectangle EBG together with the square of EB is equal (to the rectangle GEB (3. 2.), viz;) to the rectangle CED. Therefore the rectangle AFB together with the square of EF is equal to the rectangle CED.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Appendix to Book 1. Problems.

Prop 26. (22. E. 1.)

1. To constitute a triangle whose sides shall be equal to three given right lines; but it is necessary that any two of these right lines be greater than the third.

Let AB, BC, CD, be the three given right lines, situate in one right line AD, of which any two are together greater than the third (15.1). It is required to form a triangle whose sides shall successively be equal to AB, BC, CD.

Round the centre B with the distance BA describe the circle AEF, and round the centre C with the distance CD describe the circle DGH, the first meeting AD again in F, the latter meeting AD again in H. Because AB, BC together are greater than CD or CH, therefore the point A in the circle AEF is without the circle DGH; and also because AB, CD together are greater than AB or BC, take away AB or BF which is common, and CD will be greater than CF. Wherefore another point F in the circle AEF will be within the circle DGH. The two circles AEF, DGH do therefore fall partly without and partly within each other, and consequently must meet in two points external to AD, the right line joining their centres (1. Cor. 4. 3.). Let the point I be one of these, and join BI, CI, which shall the triangle required.

Fig.

Because B is the centre of the circle AEF, BI is equal to BA, and because C is the centre of the circle DGH, CI is equal to CD (13. Def. 1); also CD the remaining side of the triangle BK is the remaining given right line. Therefore BK is the triangle required.

Cor. 1. Hence upon a given right line an equilateral triangle may be constituted. (1. E. 1.)

Let BC be the given right line. Round B, C, as centres 2.
describe two circles AEC, DFB, meeting the right line BC, each, again in A, D. Then because AB, BC, CD are three right lines whereof any two are together greater than the third, because double thereto, the two circles must meet as in a point I, without BC, and BI, CI being joined, BK will be the equilateral triangle required, and the Cor. will be only a case of the Prop.

Cor. 2. Upon a given right line to constitute an isosceles triangle, each of whose equal sides shall be equal to another given right line; but it is required, that the right line to which each of the sides must be equal be greater than half the other side, which is to be the base.

Let AB, BC be the two given right lines, of which BC 3.
bisected in E, is that on which the isosceles triangle is required to be constituted. Let AE be produced to D, so that ED be equal to AE. Then BE being also equal to EC, AB

will be equal to CD, and being each greater than BE or EC, the two together AB, CD will be greater than BC, while AB, BC are greater than CD or AD. Therefore the Cor. is only a case of the Prop., and as in the Prop. the triangle BIC may be constituted, having each of its sides AI, BI equal to AB or CD.

P. 27. (2. E. 1.)

From a given point to draw a right line equal to a given right line.

4\. Let A be the given point, BC the given right line; it is required to draw from the point A a right line equal to BC.

Join AB, on which constitute the equilateral triangle ADB. Round B with the distance BC describe a circle meeting DB in G, and round D with the distance DG describe another circle meeting DA in L; I say, AL shall be equal to BC.

Because from the property of a circle DL is equal to DG, and because ADB is an equilateral triangle, AD is equal to BD, the remainder or whole AL is equal to the remainder or whole BG. But also from the property of a circle, BG is equal to BC, therefore AL is equal to BC, and it is drawn from the point A.

Cor (3. E. 1.) From the greater of two given right lines to cut off a part equal to the less.

Let AB be the greater, and C the less. At the point A draw 5. a right line AD, equal to C, and round the centre A with the distance AD describe a circle cutting AB in E, and the thing shall be done.

For AE is equal to AD because of the circle, and AD is equal to C. Therefore &c.

P. 28 (11 & 12. E. 1.)

Thro' a given point to draw a right line perpendicular to a right line given in position.

Let BC be a right line given in position, and A the given 6.7. point, either in BC, or without it, thro' which it is required to draw a right line perpendicular to BC.

Round A as a centre describe a circle with any distance, which may cut BC in two points in E, F. On EF constitute the equilateral or any isosceles triangle EFG, join ~~BC~~ GA meeting BC in H, and the thing is done.

If A the point given be in BC, the points A, H are one and the same, but if the point A be without BC, join AE, AF. Because the three sides of the triangle AEG are respectively equal to the three sides of the triangle AFG, the correspondent angles AGE, AGF, viz; HGE, HGF are

equal between themselves (6.1.). Therefore the right line GH, being drawn from the vertical angle G of the isosceles triangle EGF, to bisect the angle, will meet the base EF or BC at right angles (13.1.)

Cor. 1. To bisect a right line given in position and magnitude (10.E.1.).

7. Let EF be the given finite right line on which constitute two equilateral or isosceles triangles EAF, EGF; join AG meeting EF in H, and the thing is done.

The demonstration as well as the construction is the same as in the prop. For the vertical angle EGF of the isosceles triangle EGF being bisected by AG or GH, EF will be bisected in H (13.1.).

Cor. 2. (9.E.1.) To bisect a given rectilineal angle.

8. Let BAC be the given right-lined angle.

Round A with any distance describe a circle cutting AB, AC in E, F, which being joined, on EF describe an equilateral or isosceles triangle EGF. Join AG, and the thing is done.

As the construction little differs, the demonstration is the same as in the prop. For in the same words is the angle EAG or BAG shewn to be equal to the angle FAG or CAG.

Schol. All that is represented to be drawn in this Prop. and its Cor. is not necessary in the construction, though necessary

to the demonstration. The simple construction of the prop. is thus.

Round A with the same distance describe two small arches, cutting BC in E, F; and round E, F, with any but the same distance, describe any two arches, cutting each other in G, a point without BC on the side opposite to A. Join AG, meeting BC in H, and the thing is done.

The same simplicity of construction is applied to the corollaries.

P. 29. (23. E. 1.)

At a given point in a given right line to draw a right line making therewith an angle equal to a given right-lined angle.

AB being a given right line, and A a given point therein, it is required to draw from A a right line making with AB an angle equal to a given right-lined angle DCE. 9.

Assume any two points D, E, in CD, CE, and join DE. At the point A and on AB as a base constitute a triangle AFG, whose sides AF, AG, FG shall be equal to CD, CE, DE, each to each (26.1.). I say, the angle FAG will be equal to the angle DCE.

For the three sides of the one being equal to the three sides of the other, each to each, the triangle FAE will be

equal to the triangle DCE, and the angle FAG equal to the correspondent angle DCE (6. I.).

Schol. In the construction of this problem, no more is necessary than to describe an arch of a circle with any distance round C as a centre, cutting CD, CE in D, E; and with the same distance round A as a centre to describe an indefinite arch cutting AB in F; lastly, round F as a centre with a distance equal to DE to describe another cutting the last arch in G. Join AG, and the thing is done.

P. 30. (31. E. I.).

Thro' a given point to draw a right line parallel to a given right line.

10\. Let A be the given point, BC the given right line. Draw AD perpendicular to BC (28. I.), and AF perpendicular to AD, which shall be the parallel required.

For AF, BC being perpendicular to the same right line AD, are parallel (28. Def. I.).

Otherwise.

11\. To any point D in BC draw AD. At the point A draw AE, making with AD and towards alternate parts, the angle DAE equal to the angle ADC (29. I.). AE shall be the parallel required. — For the alternate angles DAE, ADC being equal, the right line AE is parallel to BC (9. I.).

To find a triangle equal to a given right-lined figure.

Case 1. When the given right-lined figure is a quadrilateral.

Let ABCD be the given quadrilateral. 12.

Join the diagonal BD, and parallel thereto draw CE meeting AB in E. I say, that the triangle ADE shall be equal to the quadrilateral ABCD.

The triangle BED is equal to the triangle BCD, because each stand upon the same base BD, and are between the same parallels (20.1.). Add the common triangle ABD, and the triangle ADE will be equal to the quadrilateral ABCD.

Case 2. When the given figure is of five sides.

Let ABCDE be the given five-sided figure. 13.

Join the diagonals AD, BD, and parallel thereto draw CF, EG meeting AB in F, G, Join DF, DG. Because of the parallels CF, BD, and EG, AD, the triangle ABF is equal to the triangle DBC, and the triangle DAG is equal to the triangle DAE (20.1.); therefore adding the common triangle ADB, the whole triangle DFG will be equal to the whole figure ABCDE.

Case 3. When the given figure is of six sides.

Let ABCDEF be the given six-sided figure. 14.

Draw the diagonal BD, and parallel thereto draw CG meeting AB in G. Join DG. Then the triangles DBG, DBC being

equal, the five-sided figure AGDEF will be equal to the figure ABCDEF. Wherefore by case 2. find the triangle EHI equal to the five-sided figure AGDEF, and the thing is done.

And thus by induction, each succeeding case is reduced to the preceding, and therefore a triangle may be found equal to any given right-lined figure.

Cor. 1. Hence a triangle may be found, which shall be equal to a given right-lined figure, and have one of its angles equal to a given right-lined angle.

15. By this problem, find a triangle ABC equal to the given right-lined figure, thro' C draw CD parallel to AB, and AE making with AB the angle EAB equal to the given angle, and meeting CD in E. Join BE, and ABE shall be the triangle required.

For the triangle ABE is equal to the triangle ABC (20.1), viz; to the given right-lined figure, and one of its angles EAB is equal to the given angle.

P. 32. (42. E. 1.)

To find a parallelogram which shall be equal to a given right-lined figure, and have one of its angles equal to a given one.

15. Let X be the given right-lined figure, and Z the given

right-lined angle.

Find the triangle ABC equal to X (31.1), bisect AB in E, draw CD parallel to AB, and EF making with EB the angle BEF equal to Z, and meeting CD in F; compleat the parallelogram EG; which shall be the parallelogram required.

Because the triangle ABC is between the same parallels with the parallelogram EG, but its base AB is double to the base EB of the parallelogram, the parallelogram EG will be equal to the triangle ABC (Cor. 21.1), viz; to X, and its angle BEF is equal to the given angle Z.

P. 33. (44. E. 1).

A right line being given in position and magnitude, to apply thereto a parallelogram, which shall be equal to a given right-lined figure, and have one of its angles equal to a given angle.

Let AB be the given right line, X the given right-lined figure, and Z the given angle. Draw BC making with AB the angle ABI equal to the angle Z, and IB being produced find the parallelogram BCDE equal to X, and with the given angle CBE equal to ABI. Compleat the parallelogram ABCF, join FB meeting DE in G, and compleat the ~~...~~ parallelogram FDGH, and let CB meet GH in I. I say that ABIH shall be the parallelogram required.

For it is applied to the given right line AB, and has its angle ABI equal to the given angle Z. Also the parallelograms AI, CD being complements of the parallelogram HD, are equal between themselves (23.1.). Therefore the parallelogram ABIH is equal to the parallelogram BCDE, viz; to the given right-lined figure X.

P. 34.

Upon a right line given in position to constitute a parallelogram two of whose adjacent sides shall be equal to two right lines given, and have the angle contained by them equal to a given angle.

17. Let AB be the right line given in position; X, Y, the two right lines given in magnitude, and Z the given angle.

From AB take AC equal to X (Cor. 27.1.), draw AD making with AC the angle CAD equal to the given Z (29.1.), from AD take AE equal to Y, and compleat the parallelogram AEFC, which shall be the one required.

For the adjacent sides AC, AE are equal to the two given right lines X, Y; it is constituted on the right line AB given in position, and its angle CAE is equal to the given angle Z.

18. Schol. If the angle Z be right, the parallelogram becomes a

19. 20. rectangle; if the right lines X, Y, be equal between themselves

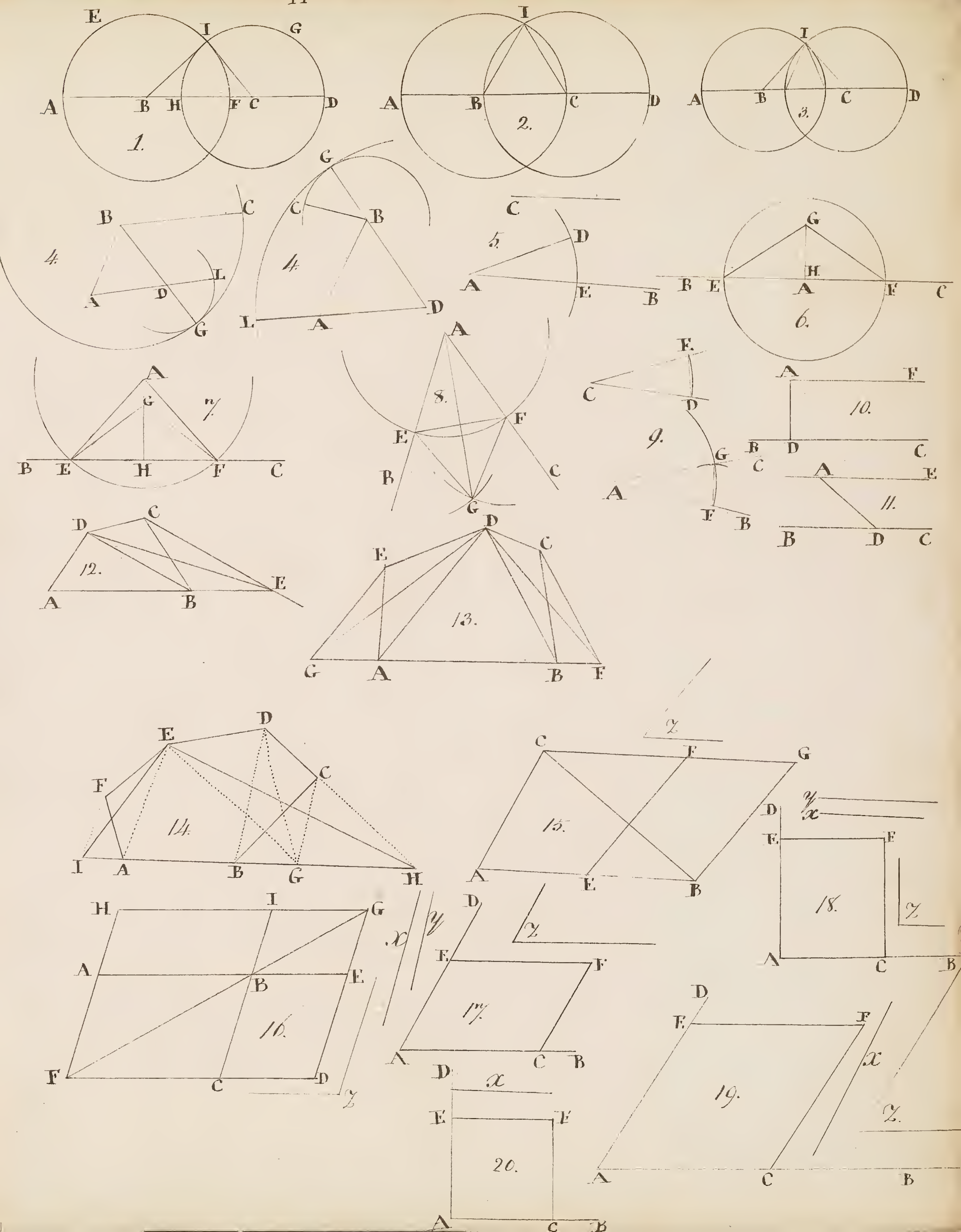
the parallelogram becomes a Rhombus, and if the angle T be also right, it becomes a square.

That all the angles of a parallelogram are right, if one be right, is comprehended in the very ~~cons~~ character of a parallelogram.

For AE falling upon the parallels AC, EB, the angles A, E are together equal to two right angles (8.1.), and therefore if the angle A be right, the angle E must also be right. For the same reason the angles C and F must also be right.

Cor. Hence it is obvious, how to constitute a square upon a given right line.

For the given angle being here a right one, and the adjacent sides being required to be equal, the two given right line are each represented by the one given right line, and the problem is only a simple case of the general one (46, F. 1.).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P. 14. (47. E. 1.)

In a right-angled triangle, the square of the hypothenuse is equal to the squares of the sides about the right angle.

1\. Let ABC be a triangle right-angled at C; I say, that the square of AB the hypothenuse is equal to the squares of AC, CB, the sides about the right angle.

Draw CD perpendicular to AB. The rectangle ABD is equal to the square of BC, and the rectangle BAD to the square of AC (Cor. 23. 1.); therefore the rectangles ABD, BAD are together equal to the squares of BC, AC together. But the square of AB is equal to the rectangles ABD, BAD; therefore the square of AB is equal to the squares of BC, AC.

P. 15.

To find a square equal to a given rectangle.

2\. Let AD, DB be the sides of the rectangle given, placed in one right line. As the problem supposes these sides to be unequal, bisect AB in C, round which point as a centre with the distance CA or CB describe a circle, and draw DE at right angles to AB meeting the circumference in E, ED will be the side of the square required.

Join CE. The squares of CD, DE together are equal to the

Fig.

square of CE (14.2.) viz; to the square of AC. But because AB is bisected in C, and unequally divided in D, the square of CD together with the rectangle ADB will be also equal to the square of AC (6.2.). Therefore the squares of CD, ED will be equal to the square of CD together with the rectangle ADB, and taking away the common square of CD, the square of ED will be equal to the rectangle ADB.

Cor. If from any point in a semicircle a perpendicular be drawn to the diameter the square of that perpendicular will be equal to the rectangle under the segments of the diameter.

P. 16.

To divide a given right line into two parts, such that the rectangle under the parts may be equal to the square of another given right line, or to the rectangle under two other given right lines.

Case 1. When the rectangle under the segments is required to be equal to the square of a given right line.

Let AB be the right line required to be divided, and X the side of the square 3

Bisect AB in C. If X be equal to AC, the thing is done. But if not equal, X must be less than AC, because the rectangle under the segments of a right line can never be greater than

Fig.

the square of half the line (6.2.). Draw CEH perpendicular to AC, in which take CE equal to X, and CH to AC or CB. Thro' E draw FG parallel to AB, and round C as a centre with the distance CH describe the circle AHB. Because CH is greater than CE, this circle will cut FG in two points I, i, from either of which draw ID perpendicular to AB. The rectangle under the segments AD, DB shall be equal to the square of ID (Cor. 15.2.) But ID is equal to CE (19.1.), viz; to X, therefore the rectangle under AD, DB is equal to the square of X, and the thing required is done.

Case 2. When the rectangle under the segments is required to be equal to the rectangle under two given right lines.

4 AB being the same as above, and bisected in C, let the two given right lines be EB, BF, placed in one right line at right angles to AB. If EB be equal to BF, the rectangle EBF is the same with the square of EB or BF, and this case becomes the same with the preceding. EB, BF being therefore unequal, bisect EF in L, and draw LO, CO parallel to AB, EF, meeting in O. Join AO, BO. Because in the triangle AOB, OC perpendicular to the base AB, bisects it at the same time, AO will be equal to BO (Cor. 13.1.) Round O as a centre with the distance OA or OB describe the circle OKB. Draw EP parallel to AB meeting OC in P. If EL be equal to AO, viz;

if OP be equal to AO, then the point P is in the circumference, and PO being produced to meet it again in R, PR will be a diameter, to which is drawn from the circumference the perpendicular AC. Therefore the square of AC or the rectangle ACB is equal to the rectangle PCR (Cor. 15.2.). But OP, LE being equal, ~~is equal to the rectangle~~ their doubles PR, EF will be equal, and therefore CP being equal to BE, CR will be equal to BF, and the rectangle PCR be equal to the rectangle EBF. Wherefore the rectangle ACB is also equal to the rectangle EBF; viz; AB is divided in C, as required, and, in this case, the right line AB will be bisected.

Now let EL be less than OA, viz; OP be less than the semidiameter of the circle. EI will then meet the circumference in two points G, g, from either of which draw GD perpendicular to AB. I say, AB is divided in D as required. 4.

Produce GD meeting the circumference again in H, and OL in I. GH is double to GI (1.3.), and also EF is double to EL. but GI is equal to EL; therefore GH is equal to EF, and the part GD being equal to the part EB the remainder DH will be equal to the remainder BF, and the rectangle GDH be equal to the rectangle EBF. Join OD, OG. In the triangles AOD, GOD are drawn from an angle in each OC, OI, perpendicular to the opposite sides AD, GD. Therefore, in

the first, the square of OD together with twice the rectangle CAD is equal to the squares of AO, AD (9.2.). But AB being double to AC, the rectangle BAD is equal to twice the rectangle CAD; therefore the square of OD together with the rectangle BAD is equal to the squares of AO, AD, and taking away the common square of AD, the square of OD together with the rectangle ADB (3.2.) will be equal to the square of AO. For the same reason in the triangle GOD is the square of OD together with the rectangle GDH equal to the square of OG. But AO is equal to OG; therefore the square of OD together with the rectangle ADB is equal to the square of OD together with the rectangle GDH; and taking away the common square of OD, the rectangle ADB will be equal to the rectangle GDH, viz; to the rectangle EBF.

Cor. 1. In the 1 Case the side of the given square must not be greater than half the given right line which is to be divided. If it be equal thereto, the solution is singular; if less, the problem admits of two solutions.

Cor. 2. In the second case, the square of half the sum of the two given right lines, containing the given rectangle, must not be greater than the sum of the squares of half the right line to be divided, and of half the difference of the two given right lines.

If equal thereto the solution is singular; if less the problem admits of two solutions

This limit appears in the demonstration, for when EL half the sum of EB, BF is equal to AO, the solution is singular, and AB is then bisected. But EL being equal to AO, the square of EL is equal to the square of AO, viz; to the squares of AC, CO, or of AC, BL, and BL is equal to half the difference of EB, BF.

Schol. The second case might have been in one step reduced to the first, by finding the side of a square which is equal to the rectangle under the two given right lines. (152.)

P. 17.

To find a point in the continuation of a given right line, such that the rectangle under the segments intercepted between the point assumed and the terms of the given right line may be equal to the square of a given right line, or to the rectangle under two other given right lines.

Case 1. When the rectangle under the segments is required to be equal to the square of a given right line.

Let AB be the given right line in which the point is to be assumed, and X be the side of the given square. 6.

Bisect AB in C, draw BE perpendicular to AB, and equal to X. Join CE and round C as a centre with the

distance CE describe a circle meeting AB in D, d. Either of these points shall answer the condition of the problem.

Because CE is greater than CB or CA, the points D, d, are each beyond the terms B, A; that is, they are each in AB produced. And because CB is equal to CA, and CD to Cd, the remainder BD will be equal to the remainder Ad, and consequently AD be equal to Bd, and the rectangle ADB be equal to the rectangle dBD. But the rectangle dBD is equal to the square of BE (Cor. 15.2.) therefore the rectangle ADB is also equal to the square of BE; viz; to the square of X.

Case II. When the rectangle under the segments is required to be equal to the rectangle under two given right lines.

7. Let EB, BF, in one right line, at right angles to AB, be the two given right lines whose rectangle is required to be equal to the rectangle under the segments.

AB being, as before, bisected in C, bisect EF also in G, draw CO, GO perpendicular to AB, EF, meeting in O, round which point as a centre, with the distance OE or OF describe a circle cutting AB in D, d. Either of these points will answer the conditions of the problem.

Because OB being joined is less than OE (4.1.), the

circle will cut AB in two points without the terms A, B; and because dD, EF in a circle cut each other at right angles, the rectangle dBD will be equal to the rectangle EBF (3. Cor. 16. 2.). But dD, AB, being each bisected in C, dA will be equal to BD, dB to AD, and the rectangle dBD to the rectangle ADB. Therefore the rectangle ADB is also equal to the rectangle EBF.

Schol. These three problems are in the highest estimation with the ancient Geometers, not only ~~in~~ from their extensive application in the solution of many other problems, but probably by their characteristic affinity to a discriminating and essential property of the three principal sections of the Cone.

These three problems are thus differently enunciated by the Ancients.

I. (15. 2.). To apply a rectangle to a given right line, which shall be equal to a given right lined figure, and neither exceed nor be deficient.

II (16. 2.). To apply a rectangle to a given right line, which shall be equal to a given square or rectangle, and be deficient by a square.

III. (17. 2.). To apply a rectangle to a given right line, which shall be equal to a given square or rectangle, and exceed by a square.

The meaning of this enunciation and its co-incidence with the enunciation as expressed in the problems above, will be immediately perceived by a reference to the proper figures, with their explanation.

Let AB be a given right line, and X a given rectangular quadrilateral, and on AB let a rectangle be applied, viz; AE, equal to X. Then if (Fig. 8.9.10.) AE have for its base the whole of AB, the case answers to the first of these three problems, viz; the 15.2., if it have its base a part of or less than AB, the case answers to the 2nd, viz; to the 16.2.; if it have for its base more than AB, viz; AB continued to D, the case answers to the third, or to the 17.2.

8. In the 1st the rectangle AE applied to AB is neither deficient of, nor exceeds the rectangle applied to the whole of AB.

9. In the 2nd the rectangle AE applied to AB, rests upon AD, and is deficient of the rectangle AG resting upon the whole AB as its base, by the rectangle BE, which is a square. For both enunciations agree in the rectangle ADB being equal to X, and AE being this rectangle actually applied, the side DE of this rectangle is equal to DB the deficiency whereby AD is short of AB, and therefore BE is a square.

10. The 3d exhibits the rectangle AE applied to AB standing

 Fig.

upon its base AD, greater than AB by the excess BD, and the rectangle AE exceeds the rectangle resting upon AB absolutely, as its base, by the rectangle BE, which for the same reason as in the 2nd case, is a square. This answers to the 17.2.

P. 18. (11. E. 2.).

To divide a given right line into two parts, such that the rectangle under the whole and one part may be equal to the square of the other part.

Let AB be the given right line. – Draw AC perpendicular & 11. equal to AB, bisect AC in E, join BE, and produce EA to F, making EF equal to EB; also from AB take AH equal to AF; I say that AB is cut in H as required.

Compleat the square ACDB, and drawing FG parallel to AB, thro' H draw a parallel to AC, meeting CD in K, and FG in G. Because AH is equal to AF, and the angle at A is right, the parallelogram AG is a square (Cor. 34.1.), and CG is a rectangle contained under CF, FG, viz; under CF, FA. But because AC is bisected in E, and a point F is assumed in CA produced, the rectangle under CF, FA together with the square of AE is equal to the square of EF (6.2.), viz; to the square of EB. Therefore the rectangle CG together with the square of AE is equal to the square of EB. But the square of EB is equal to the squares of AB, AE (14.2.), that is

to the square AD together with the square of AE. Wherefore the rectangle CG together with the square of AE is equal to the square AD together with the square of AE, and taking a=:way the common square of AE, the rectangle CG will be e=:qual to the square AD. Lastly, taking away the com=:mon rectangle CH, the remaining rectangle AG will be equal to the remaining rectangle HD. But it has ap=:peared that AG is a square, viz; described upon AH, and because AD is a square, the rectangle HD is the same with the rectangle under AB, BH. Therefore the rectangle under AB, BH is equal to the square of AH, and ∴ AB is divided in H as required.

P. 19.

To produce a given right line to a point, such that the square of the right line given shall be equal to the rectangle under the whole extended line and the added part.

Bisect the right line given, viz; AB, in C, draw BE perpendi=:cular to AB, and equal to AB; join CE, and round C as a cen=:tre with the distance CE, describe a circle cutting AB in D, d. Either of these points shall answer the conditions of the problem. — Because Cd is equal to CD, and CA to CB, dA will be equal to BD, and dB to AD, and the rectangle dBD to the rectangle ADB. But the rectangle dBD is equal to the square

of BE (Cor. 15.), viz; to the square of AB. Therefore also the rect=:angle ADB is equal to the square of AB.

Schol. This is merely a case of the 17. of this.

P. 20.

To find a right line, whose square shall be equal to a given right-lined figure.

Let X be the given right-lined figure. Find the right-an:gled parallelogram ABDE, equal to X (32.1.) If the adjacent sides AB, BD be equal between themselves, AD is a square, and what was required is effected. But if not, produce AB to F, making BF equal to BD. Bisect BF in C, round which point as a centre, with the distance CA or CF, des:cribe a circle cutting DB in G. I say that BG is the right line required. 14.

Join CG. Because AF is bisected in C, and unequally divid:ed in B, the rectangle ABF together with the square of BC is equal to the square of AC or CG (6.2.) But because the angle B is right, the square of CG is also equal to the squares BG, BC (14.2.). Therefore the rectangle ABF together with the square of BC, is equal to the squares of BG, BC, and taking away the common square of BC, the rectangle ABF is equal to the square of BG; viz; the square of BG is equal to X.

Appendix to Book III.

P. 28. (1. E. 3.)

To find the centre of a given circle.

1. Let ABD be the given circle. Therein inscribe any right line AB, which bisect in E, and thro' E draw DF at right angles to AB, meeting the circle in D, F. Bisect DF in C. I say that the point C shall be the centre of the circle.

Because DF bisects at right angles the right line AB in:=scribed in the circle, the centre of the circle is in DF (1. Cor. 1. 3.); and therefore must be in the point C, which is the bisection of DF.

P. 29. (17. E. 3.)

From a given point, either in the circumference, or with:=out the circle, to draw a right line which shall touch a given circle.

Let BCG be a given circle, A a point given either in the cir:=cumference, or without the circle, from which it is required to draw a right line touching the circle.

2. First, if A be in the circle, draw the diameter AB, and AD at right angles thereto. AD will touch the circle in A (11. 3.). But if the point be without the circle, draw as before a right line thro' A, and the centre C. Bisect AC in D, and

round D as a centre with the distance DC or DA, describe a circle cutting the circle BEG in E. Join CE, AE. Because AEC is an angle in a semicircle, AE is at right angles to CE, a semidiameter of the circle BGE (12.3.), and therefore as before, AE will touch the circle in E.

The 2nd Case, otherwise.

Join AC, meeting the circle BEG in B, round C with the distance CA describe the circle AFH, and draw BF perpendicular to AB, meeting this circle in F. Join CF, meeting the given circle BEG in E; I say, that AE, being joined, shall touch the circle in E.— Because of the circles, the two sides CB, CF of the triangle CBF are equal to the two sides CE, CA of the triangle CEA, and the angle BCF contained by the one two is the same with the angle ECA contained by the other two; therefore the angle CBF is equal to the correspondent angle CEA. But the angle CBF is right, therefore CEA is also a right angle, and so AE being at right angles to the diameter EC, touches the circle in E. 4

P. 30. (25. E. 3.).

A segment of a circle being given, to compleat the circle of which it is the segment.

Let ADB be the given segment. 5.

Join AB, and bisect it in E. Draw ED perpendicular to AB, meeting the circumference in D, join AD, and draw AC making with AD the angle DAC equal to the angle ADE (29.1.). Because the angle ADE of the right-angled triangle AED is less than a right angle, the two angles ADE, DAC are together less than two right angles, and therefore DE, AC must meet. Let them meet in C, & join BC. Round C with the distance CA describe a circle. This shall be the circle required.

Because the angles ADC, DAC, are equal, the sides DC, AC are also equal (2. Cor. 4.1.), and because AE is equal to EB, EC common, and the angles AEC, BEC are equal, being each right, the sides AC, BC are also equal (3.1.). Therefore the circle described round C with the distance CA, passes thro' D, B, and therefore is the circle of which ADB is the segment (3. Cor. 3.3.).

Cor. (20. E. 3.). In this problem is included the bisection of a given circumference.

6. For ADB being the given circumference, and every thing remaining as in the problem if the circumference ADB be a semi-circle, the point E becomes the same with the centre C, and therefore the angles ACD, BCD, being equal, because right are subtended by equal

 Fig.

segments AD, BD (Cor. 14.3.), that is, the circumference ADB is bisected in D.

But if the centre be not in AB, then still ED being drawn as 5.
in the problem, the circumference ADB will be bisected in D.—
For every thing remaining as in the problem, by the same argument that proves the side AC to be equal to the side BC, is the angle ACE or ACD inferred to be equal to the angle BCE or BCD, and therefore the arch AD to be equal to the arch BD (Cor. 14.3.), that is, the circumference ADB to be bisected in D.

P. 31. (34. E. 3.)

At a given point in a given circle to draw a right line which shall cut off a segment containing an angle equal to a given angle.

Let ABC be the given circle, A the given point therein, 7.
and X the given angle. It is required to draw from A a right line which shall cut off a segment containing an angle equal to X.

Thro' A draw the right line AE touching the circle in A (29.3.), and AB making with AE the angle BAE equal to the angle X, and the thing shall be done.

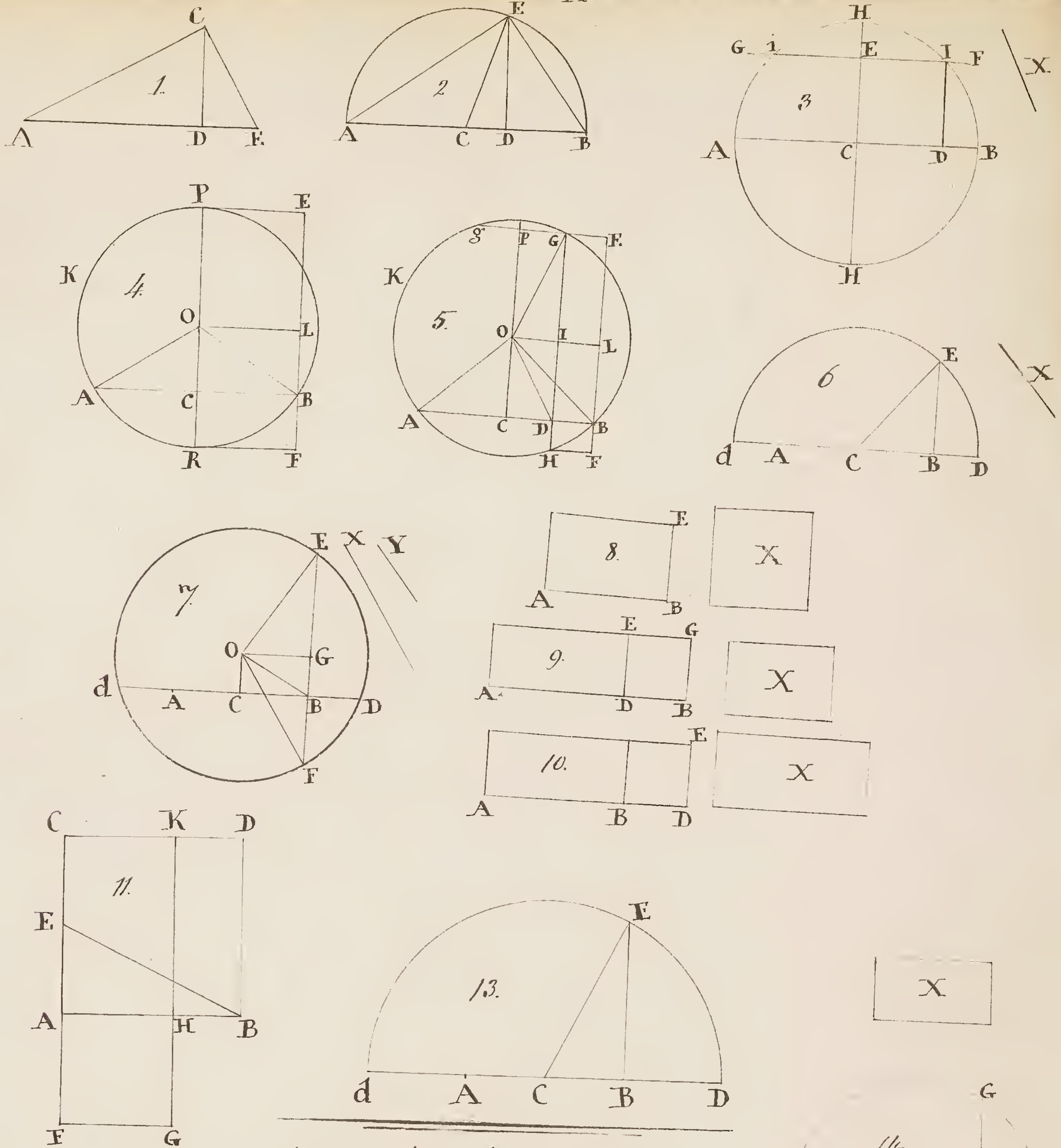
Because AE touches the circle in A, from which point is drawn AB cutting the circle into the two segments ACB, AFB, the segment ACB shall contain an angle,

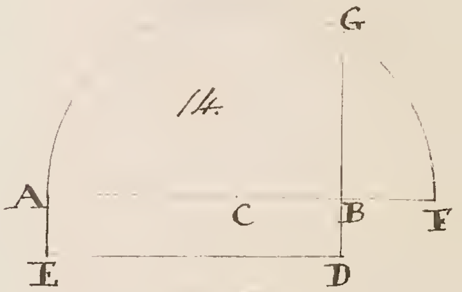
equal to the alternate angle BAE, made by AB with the touching line AE (16.3.), viz; to X.

P. 32. (33. E. 3.)

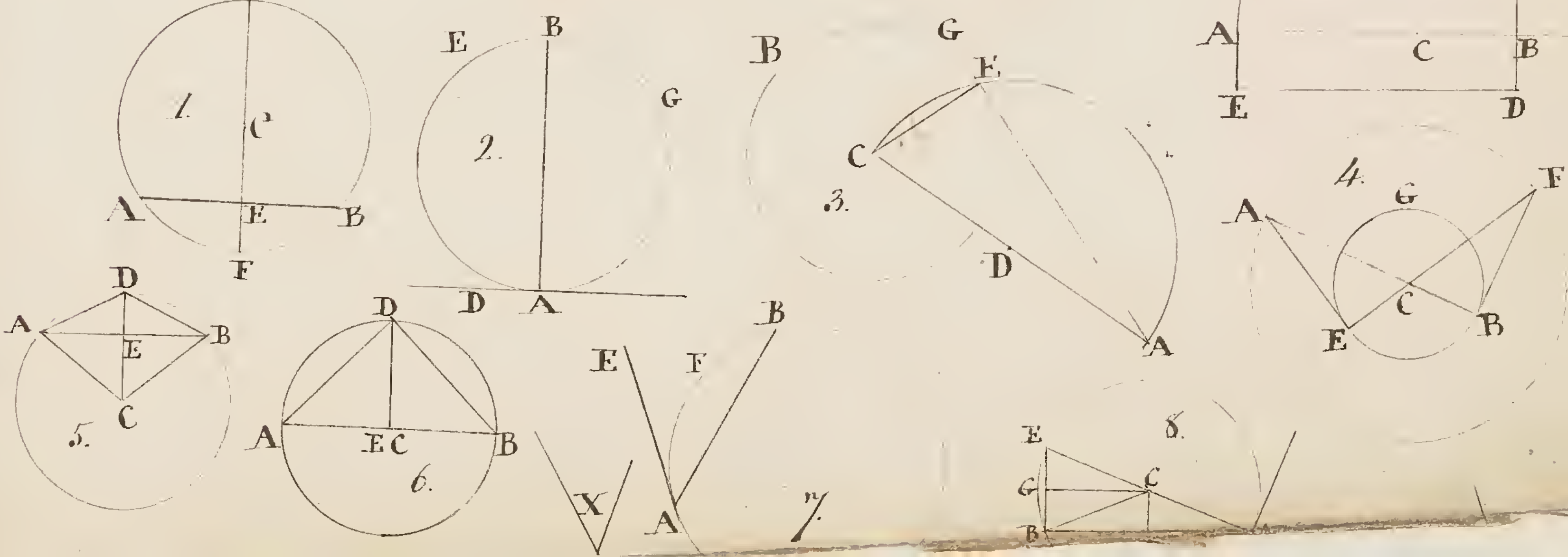
Upon a given right line to describe a segment of a circle containing an angle equal to a given angle.

8. Let AB be the given right line, and X the given angle. — Draw AD making with AB the angle BAD equal to X (29.3.), and draw also AE perpendicular to AD, and BE perpendicular to AB, meeting each other in E, bisect AE in C; I say, that a circle described round the centre C with the distance CE or CA will be the one required, viz; it will pass thro' B, and the segment AEB will contain an angle equal to X. Draw CF parallel to BE, and CG to AB. Then because of the parallels, the angles ECG, CEG are respectively equal to the angles CAF, ACF, and EC is equal to AC. Therefore EG is equal to CF, and also BG is equal to CF. Therefore EG is equal to BG, and CG being common, and the angles at G right, EC will be equal to BC. Consequently the circle described round C passes thro' B, and therefore the angle AEB in the segment AEB is equal to the angle BAD, viz; to the angle X.





Appendix to Book III.



Book IV.

Definitions.

1. A Regular right-lined figure is that, whose sides and angles are all equal.
2. A regular figure is said to be inscribed in a similar regular figure, when all the angles of the inscribed ar in the sides of the circumscribing one, each in each. And, vice versâ, to Circumscribe, when each side of the circumscribing one has in it an angle of the inscribed.
3. A regular figure is said to be inscribed in a circle, when all the angles of the figure are in the circumference of the circle; and therefore a Circle is said to Circumscribe a regular figure, when the circumference passes thro' each angle of the figure.
4. A circle is said to be inscribed in a regular figure, when the circle touches each side of the figure; and therefore a regular figure to Circumscribe a circle when every side of the figure touches the inscribed circle.
5. A right line is said to be Inscribed in a circle, when each extremity of the line is in the circumference.

P. 1. (1. E. 4.)

To inscribe in a given circle and at a given point a right line equal to a given right line, provided that the

given right line be not greater than the diameter.

At a given point A it is required to inscribe in a given circle BIL a right line equal to a given right line Z; but it is necessary that Z be not greater than the diameter of the circle.

If the given right line Z be equal to the diameter, draw a right line thro' A, and C the centre of the circle, and the thing is done. 1.

But if Z be less than the diameter, then

1. Let the point A be given in the circumference of the circle BIL. 2.

Draw the diameter AB, and from AB the greater cut off AE equal to Z the less, round A as a centre with the distance AE describe a circle meeting the given circle BIL in D. Join AD which shall be the right line required to be inscribed.

For AD is equal to AE, viz; to Z, and is inscribed in the circle BIL.

II. When the point is given either, without or within the circle BIL. 3.4.

Thro' C the centre of the circle and A, draw the diameter EB. At the point E inscribe EK equal to Z (by Case I), & round C as a centre describe a circle passing thro' A, and

meeting ED, produced if necessary, in F. Join CF meeting the circle BIL in G, also join AG meeting the circle BIL again in H. I say, that GH, which is drawn from the point A, and is inscribed in the circle BIL, shall be equal to Z.

Join CH, CD. Because CA is equal to CF, CG to CE, and the angle ACG to the angle FCE, as being either the same or vertical to it, AG will be equal to FE, and the angle CAG equal to the angle CFE (3.1.). Again, because AC, CH, are equal to FC, CD, each to each, and the angle CAH equal to the angle CFD, AH will be equal to FD (12.1.). Wherefore AG being equal to FE, and AH equal to FD, the remainder HG will be equal to ED.

Observe that on account of the isosceles triangles CGH, CED, each of the angles AHC, FDC, are greater than a right angle.

P. 2. (2. E. 4.).

In a given circle to inscribe a triangle equiangular to a given triangle.

ABC being the given circle, and DEF the given triangle, at any point A in the circumference of ABC draw GAH touching the circle in A; also AB, AC making with GH the angles GAB, HAC equal to the angles DEF, DFE, and meeting the circumference in B, C. Join BC, and ABC shall be the triangle required.

Because CH touches the circle in A, the angle GAB is equal

to the angle ACB, and the angle HAC to the angle ABC, the angles in the alternate segments (16. 3.). But the angles GAB, HAC, are equal to the angles DEF, DFE; therefore the angles ACB, ABC, are equal to the angles DEF, DFE, and consequently the remaining angle CAB is equal to the remaining angle EDF (3. Cor. 10. 1.).

P. 3 (3. E. 4.)

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle. 6.

Produce EF both ways to G, H. From the centre O of the circle draw OA to the circumference, and OB, OC making with AO the angles AOB, AOC, equal to the angles DEG, DFH. Draw KAL, KBI, LCI at right angles to OA, OB, OC. Because the four angles of a quadriteral are equal to four right angles (7. Cor. 10.1), and the two angles A, B, of the quadrilateral AOBK are right, the two remaining angles at O, K are together equal to two right angles. But the two angles DEG, DEF are also equal to two right angles (1.1.), and the angle AOB is equal to the angle DEG; therefore the angle at K, or IKL is equal to the angle DEF. For the same reason is the angle ILK equal to the angle DFE, and consequently the remaining angle KIL of the triangle

∴KL is equal to the remaining angle EDF of the triangle DEF (3. Cor. 10.1.). A triangle is therefore constituted about the given circle which is equiangular to the given triangle, and the three sides touch the circle, because they are each at right angles to a diameter at its extremity (11.3.). Therefore the triangle IKL is described about the given circle ABC, equiangular to the given triangle DEF (4. Def. 4.).

P. 4. (5. E. 4.).

To describe a circle about a given triangle.

7. ABC being the given triangle, bisect AB, AC, in D, E, and draw DF, EF at right angles to AB, AC. Join DE. Then the angles ADF, AEF being each right, the angles FDE, FED are each less than right, and together less than two right angles. Therefore DF, EF do meet. Let them meet in F. — Join AF, BF, CF. Because AD is equal to BD, DF common, & the angle ADF equal to the angle BDF, the base AF is equal to the base BF (3.1.). For the same reason is AF equal to CF; therefore FA, FB, FC are each equal to each other, & consequently a circle described round F as a centre with the distance of either of them will pass thro' the three points A, B, C; that is, will circumscribe the triangle.

Schol. 1. This problem is the same with describing a circle, which shall pass thro' three given points, which are

not in one given right line.

Schol. 2. It is manifest that as the centre falls within the triangle, without, or in one of the sides, all the angles of the triangle will be less than right angles, or the angle opposite to the side, beyond which the centre falls, will be greater, or in the third case, will be equal to, a right angle.

Fig. 8. 9.

For accordingly each angle will be in a segment greater than a semi-circle, or the angle will be in a segment less than, or equal to, a semi-circle (12.3.).

P. 5. (4. E. 4.)

To inscribe a circle in a given triangle.

Let ABD be the given triangle. 10.

Draw AC, BC bisecting the angles BAD, ABD, and meeting in C. Draw CE, CF, CG perpendicular to AB, BD, AD. Because the angle CAE is equal to the angle CAG, the angles E, G, each right, and therefore equal, and the common side AC is opposite to the equal angles at E, G, the side CE will be equal to the correspondent side CG (11.1.). For the same reason is CE equal to CF, and consequently a circle described round C with the distance CE will pass thro' F, G, also; and the three sides AB, BD, AD will each touch the circle in E, F, G, because at these points perpendicular to the

diameters CE, CF, CG (11.3.). Therefore the circle EFG is inscribed in the triangle ADB (4. Def. 4.), as required.

P. 6. (6. E. 6.).

To inscribe a square in a given circle.

11. Draw the diameters AB, DE at right angles to each other, and meeting in the centre C. Join AD, AE, BD, BE. ADBE shall be the square inscribed, as required.

Each of the angles of the quadrilateral ADBE is right, because each is an angle in a semi-circle (12.3.); and because in the triangle ABD, the right line DC drawn from the vertex D bisects the base AB at right angles, AD will be equal to BD (Cor. 13.1.). For the same reason is AD equal to AE, AE to BE, and BE equal to BD. Therefore ADBE is a square, and is inscribed in the circle given, ~~ADBE~~ ADBE.

P. 7. (7. E. 4.).

To describe a square about a given circle.

12. Let FGH be the given circle, C its centre. Draw the diameters FH, GI at right angles to each other, and AFB, EHD, parallel to IG, and AIE, BGD parallel to FH. The quadrilateral ABDE shall be a square, circumscribing the circle.

Because FH, IG are perpendicular to each other, AB, ED

parallel to IG will be perpendicular to BD, AE, which are par-
allel to FH. Therefore ABDE is a rectangle, and the diameters FH, IG, ~~HH~~ being also equal, AB, BD which on account of the parallels are equal thereto, will be equal between themselves, and for the same reason all the sides of the rectangle ABDE be equal between themselves. Therefore ABDE is a square, and its sides severally touch the circle because they are each perpendicular to a diameter at its extremity.

P. 7.

To describe an isosceles triangle which shall have each of the angles at the base double to the angle at the vertex.

Let FG be any right line which divide in H so that the rectangle under FG and the less segment GH may be equal to the square of the greater segment FH (13. 2.). Round F as a centre with the distance FH describe a circle, and drawing GI to touch it in I (29. 3.), join FI, HI. I say, FHI shall be the isosceles triangle required.

From the property of the circle it is isosceles. Also because GI touches the circle, the rectangle FGH is equal to the square of GI (20. 3.) and it is also equal to the square of FH. Therefore GI is equal to FH, viz, to FI, and the

Fig.

triangle IFG is isosceles. The angle FGI is therefore equal to the angle IFG, and the angle GIH is also equal to the angle IFG, because GI touches the circle in I (16.3.)

P. 8. (11. E. 4.)

To describe a regular pentagon in a given circle.

13. Let ABH be the given circle, C its centre; draw any diameter CCA and produce CA to D so that the square of AC is equal to the rectangle CDA (19.2.). Because the rectangle CDA is greater than the rectangle CAD (3.2.), the square of AC will be greater than the rectangle CAD, and consequently AC will be greater than AD, and therefore a circle described round D with a distance equal to AC will cut the circle ABH in two points B, F. Join AB, AF, CB, CF, DB, DF, BF. Because the rectangle CDA is equal to the square of AC, viz. of DB, the right line DB will touch the circle described round the triangle ABC in the point B (21.3.), and therefore the angle ABD will be equal to the angle ACB in the alternate segment of that circle (16.3.). But CB being equal to BD, the angle DCB or ACB is equal to the angle CDB or ADB (4.1.), therefore the angle ABD is equal to the angle ADB, and AB is equal to AD (Cor. 4.1.)

For the same reason is AF equal to AD, and therefore the angle ACF to the angle ACB (4. Cor. 14.3.). Again the angle CAB is equal to the angles ABD, ADB (10.1.), viz; is double to the angle ADB, and therefore double to the equal angle ACB. Wherefore in the isosceles triangle ACB, the angle CAB, and therefore the angle CBA also, is double to the angle ACB at the vertex. But the triangles ACB, ACF being in every respect equal, the angle CAB will be equal to the angle CAF, and the angle BCF double to the angle ACB. Also the angle EFB is equal to the angle CAB, and the angle EBF to the angle CAF, because standing upon the same arches EB, EF, & BEF equal to the angle ACB, because the same angle BCF is double to each of them (13.3.). Wherefore the triangle BEF is also isosceles, and each of the angles EBF, EFB at the base BF is double to the angle BEF at the vertex. Draw BG, FH bisecting the angles EBF, EFB, and meeting the circumference in G, H, and join FG, EG, BH, EH. I say BFGEHB is a regular pentagon inscribed in the circle ABH.

Because the angle EBF is double to the angle BEF, each of the angles FBG, EBG will be equal to the angle BEF, and therefore each of the right lines FG, EG

will be equal to B.F (2. Cor. 15.3.). For the same reason are BH, HE each equal to BF, therefore the five-sided figure inscribed in the circle BFGEHB is equilateral, and consequently is equiangular, that is, it is a regular pentagon.

Cor. If each arc subtended by a side of the pentagon be bisected, and to the points of bisection be drawn right lines from the angles of the pentagon, a regular decagon will be inscribed in the circle.

Or directly thus, produce DB, DF, cutting the circumference again in I, K; draw the diameters BL, AC, FO, KH, IG. The points I, B, A, F, K, G, L, E, O, H, will be the angular points of a regular decagon inscribed in the circle.

The demonstration is almost included in the demonstration of the proposition.

Cor. 2. (10. E. 3.). To constitute an isosceles triangle, such that each of the angles at the base shall be double to the angle at the vertex.

This is done in the Prop. as in the construction of the triangle ABC.

P. 9. (12. E. 3.)

To describe a regular pentagon about a given cir-

-cle.

Let the points A, B, D, E, F, be the angular points of a re-gular pentagon inscribed in the circle, thro' which points draw right lines touching the circle, and by their concourse with each other forming the five sided figure GHIKLG. From the centre C draw CF, CA, CB, CL, CG, and join AF, AB. Because FCA, ACB are each isosceles triangles, whose sides, being semi diameters of the same circle, are all equal between themselves, and the bases AF, AB are also equal, the angles at the bases AF, AB will all be equal also, viz; CFA, CAF, CAB, CBA; will be angles mutually equal, each to each other. But the angles CFL, CAL, CAG, CBG, being each right, are each equal to each other. Therefore the remaining angles LFA, LAF, GAB, GBA, are also equal each to each other, and the bases AF, AB, being equal, the triangles FLA, AGB are equal isosceles triangles (2. Cor. 4.1. & 11.1.) and the sides LF, LA, GA, GB will be equal, each to each other. In the same manner is it shown that GB, BH, HD, ID, IE, KE, KF, LF are all equal each to each other. Therefore their dou-bles LG, GH, HI, IK, KL, are all equal each to each other, viz; the figure GHIKLG is an equilateral and equiangu-lar pentagon. It is therefore a regular pentagon and

circumscribes the circle, because each of its sides touches the circle.

P. 10. (15. E. 4.)

To inscribe a regular Hexagon in a given circle.

14. Let ABE be the given circle, C its centre. At any point A in the circumference with the distance AC describe a circle cutting the circumference ABE in B, D. Draw the diameters AF, BG, DE, and join AB EFGDA, which shall be the hexagon required.

The triangles ACB, ACD, are from the construction equilateral, and the sides of the one equal to the sides of the other. Therefore the angle ACD, that is, the angle ECF is equal to the angle ACB. But the angle BCF is equal to the angles ABC, BAC, that is, because all the angles of the triangle ABC are equal, the angle BCF is equal to twice the angle ACB. But it has appeared that the angle ECF is equal to the angle ACB, therefore the remaining angle BCE is also equal to the angle ACB. Wherefore the angles ACB, ACD, BCE, being equal, each to each, the remaining angles at the centre, being vertical thereto, are also equal to them and to each other. All the angles at the centre therefore being equal between themselves, the right lines upon which they

stand, viz; AB, BE, EF, FG, GD, DA; are all equal (4. Cor. 14.3.), and the angles which they make with the semi-diameters being equal, the angles made by the sides thereof which are the double of the former, will also be equal. ABEFGDA is therefore a regular hexagon inscribed in the circle. ABE.

Cor. 1. If the circumferences subtended by the sides of the hexagon be each bisected, the points of bisection together with the angular points of the hexagon, will be the points of a regular duodecagon inscribed in the circle.

Cor. 2. If at the angular points of a hexagon inscribed in a circle, right lines be drawn touching the circle, they will by their concourse constitute a regular hexagon circumscribing the circle, and the demonstration is the same as in the case of a pentagon in P. 9.

P. 11.

To inscribe a regular quindecagon in a given circle. Let AEB be the given circle. Inscribe therein the equiangular triangle ABD (2. 4.), and also from the point A the regular pentagon AEFGHA. Suppose AF to be the ~~side of~~ circumference subtended by the side of a Quindecagon; then as the whole circumference contains

AI fifteen times, while it contains AB three times, and AC five times, it is manifest that AB contains AI five times, and that AC contains it three times. Therefore the circumference ACF being double to AC is equal to six times AI, and the circumference AB to five times AI, and consequently BF is equal to AI. Join BF, and inscribe it successively from the point B or F thro' the whole circumference, and the quindecagon required will be inscribed.

Cor. In the same manner as in the 2. Cor. 9. 4. may a Quindecagon be described which shall circumscribe a given circle.

P. 12.

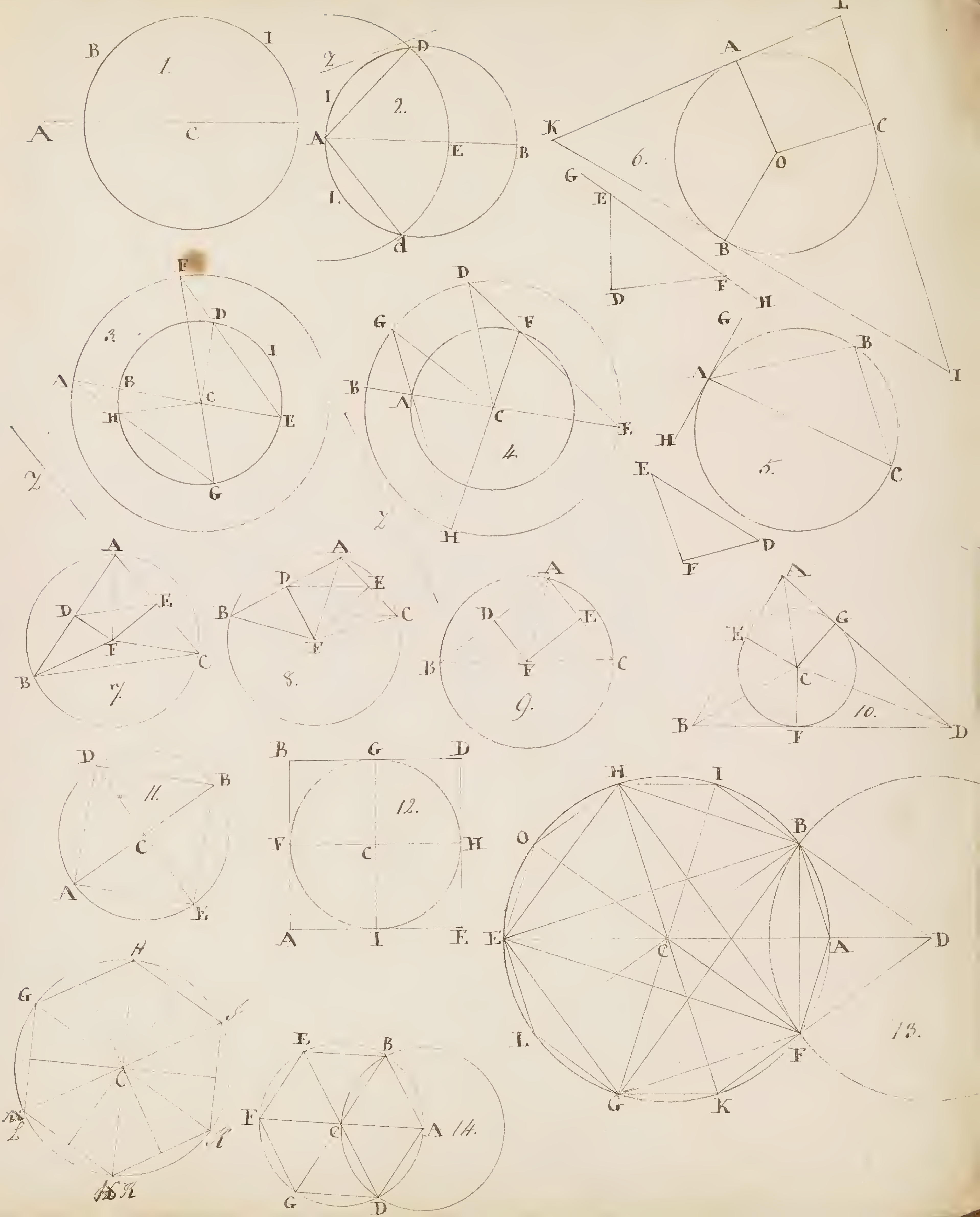
To inscribe a circle in a regular figure.

Let GHFK &c. be any regular figure, in which a circle is required to be inscribed. Draw LC, GC bisecting any two adjoining angles of the figure, viz: the angles KLG, GLH. From C the concourse of LC, GC draw CF, CA, CB perpendicular to KL, LG, GH and join CH. Because the angles KLG, LGH are equal (1. Def. 4.), their halves CLG, CGL are equal between themselves and CLG is an isosceles triangle, and CA being drawn from the vertical angle at C perpendicular to the base LG, will bisect LG in A (13. 1.). Again

because in the triangles CLG, CGH, the side CL is equal to the side CG, and the side LG to the side GH (1.Def.4.4), and the angle CLG to the angle CGH, the remaining side CG of the one is equal to the remaining side CH of the other, viz; CGH is also an isosceles triangle equiangular and equal to the triangle LCG, and therefore CB will bisect also the side GH, and be equal to CA: For the same reason does CF bisect LK, and will be equal to CA; so for the same reason will the perpendiculars from C on each of the sides of the polygon bisect each side, and be each equal to CA. Therefore a circle described round C with the distance CA will touch each side of the polygon and therefore be inscribed in it.

Cor. In the same manner may a circle be described which shall circumscribe a given regular polygon.

For the same things remaining, as all the right lines drawn from C to the angles of the figure may be proved to be equal in the same manner as CG is shewn to be equal to CL, and CH to CG, therefore a circle described round C with the distance CG will pass thro' all the angles of the figure, that is, will circumscribe it.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John Yates.

Elements of Geometry.

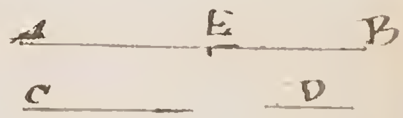
Book. V.

Definitions:—

1. Quantities of the same kind are said to have a Ratio or Proportion to each other when they are compared with respect to magnitude.

2. The comparison of magnitudes either respects the Quantity by which the one exceeds the other or the Quantity of the same equal parts which are contained in each of them.—

Thus the magnitudes A, AB & C are compared with each other as the one AB exceeds the other C by the Quantity AE; or as they are referred to a third magnitude D, which repeated a certain number of times would measure or equal each of them. In this sense are AB & C said to have a ratio or proportion to each other, When AB contains so many times a third magnitude D & C contains so many times the same magnitude.—



Note. This Latter is the kind of proportion we are now to treat of & is called Geometrical.—

3. This Common Magnitude to which two or more Magnitudes are referred is called a Common Measure of these magnitudes.—

4. One Magnitude is called an Aliquot part when repeated a certain number of times, it exactly measures, ie.

neither is deficient of nor exceeds it.....
5.. One magnitude is said to be a Multiple of another which is equal to or exactly contains so many times the other..
6. Four magnitudes are said to have the same ratio or Proportion to each other when the same multiples that the first & second are of a Common measure to which they are referr'd; the same multiples are the Third & Fourth of another common measure to which they are referred.

Thus if the Magnitudes A, C be referred to a common measure E, & two other magnitudes B, D be respectively Equimultiples of the common measure F, to which they are referred i.e. if B be the same Multiple of F that A is of E, & D be the same multiple of F that C is of E, then are A, B, C, D said to be proportional, or A to have the same Proportion to C that B has to D.....
7. Of two Magnitudes referred to a common measure the first is called the Antecedent the second the Consequent.
8. Analogy or continued proportion is when of three or more magnitudes the first has the same proportion to the second that the second has to the third that the third has to the fourth and so on continually.
In this Proportion each term is an antecedent to the one following and a consequent to that preceding.
Duplicate Ratio is when of four Magnitudes the first has the same proportion to the second that the third drawn

into itself has to the fourth drawn into itself.
In like manner it is understood of Triplicate Quadruplicate &c Ratio that the third & fourth terms are three times four times &c drawn into themselves.

10 Alternate Ratio is when the antecedent is compared with the antecedent and the consequent with the consequent.

11. Inverse Ratio is when the Consequent becomes the antecedent and the antecedent the Consequent.

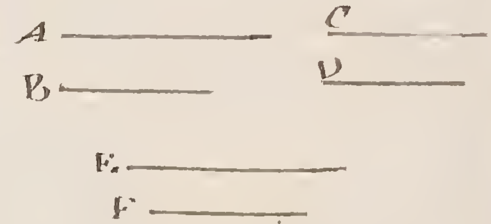
12. Compound Ratio is when each antecedent & consequent taken as one magnitude is compared with each consequent.

13. Divided Ratio is when each excess wherein the antecedent exceeds the Consequent is compared with each consequent.

14. Converse Ratio is when each antecedent is compared with each excess whereby the Antecedent exceeds the Consequent.

15. Equality of Ratio is when two magnitudes and also two others having the same proportion to two and the same magnitudes are in Proportion to each other.

Thus if A be to B as E is to F and again C be to D as E is to F then A is compared with B and C with D in the same proportion.



Axioms

1. Equimultiples of the same or equal Magnitudes are equal to each other & Contra.
2. Magnitudes are to each other as their Equimultiples.
3. Magnitudes are to each other as their like aliquot parts.

These two last Axioms are obviously derived from Def. 6.

4. Two Magnitudes to which the same Magnitudes are proportional are equal to each other.

This Axiom is sufficiently self evident since otherwise the proportion which things of the same kind have to each other would be no Limitation of Magnitude, but the truth thereof immediately appears from Def. 6.

Thus if A be to B as C is to D and also A be to B as C is to E then by Def. 6 D, E are equimultiples of one and the same common measure and therefore equal to each other (Ax 1.).

Postulates.

1. If a Multiple of one magnitude be given an equimultiple of another may be assumed. No more is necessary to this than to repeat the Magnitude so many times.
2. If an aliquot part of one Magnitude be given a like aliquot part of another may be assumed.

This is only to divide a Magnitude into the same number of equal parts into which another Magnitude of the same kind is divided a problem which is solved in the case of Right lines ( ). And Right lines may be considered as an universal Representative of Magnitude where only the Relation of Magnitude between things of the same kind is enquired into. But it sufficient to all the purposes intended by this postulate if only the possibility of the Demand may be admitted, and thence a magnitude representativeing such an aliquot part be assumed.

3. A Common measure to two magnitudes may be assumed.

The supposition that this postulate is only possible in the case of Quantities which are commensurable to each other has introduced all the obscurity which is found

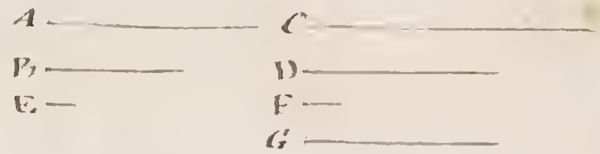
in the fifth Book of Euclid. But in Despight of all the Judgment of severe Geometers I must think that in this perspicuity has been sacrificed to Chimerical Truth.

No two Magnitudes are more strictly incommensurable to each other than the Side and the Diagonal of a Square, yet if a square was conceived to be inscribed in the great Orbit of the Earths motion round the Sun a finite determinate magnitude by a very easy process may be found which shall measure both the side & Diameter of this immense square without exceeding or being deficient of either in so much as the thickness of a Grain of Sand. And if you say that this Defect or Excess deserves regard you may by extending the process approach as much nearer to the truth as you please. How very delicate therefore must these Geometers be who will rather suffer youth to pass thro' the Elements of a Science without any Ideas at all unless derived from the very Source which they disavow than give up this fanciful nicety. It would indeed be highly ungeometrical to investigate the Properties of Proportional Magnitudes from a Definition which supposed them Multiples the one of the other because it would be inferring a General truth of all proportionals from a circumstance which obtains in very few of them. But the referring all Magnitudes whatever to some common measure is universal and the properties derived therefrom if the Supposition can be admitted will be universal also. This common measure may indeed be infinitely small but what then? It is equally Geometrical

If two magnitudes be referred to another which is the thousand millionth part of each of them or to a Magnitude which is only the Half or third part thereof.

Prop. 1.

A Common measure of two Magnitudes being given to find a like common measure of two other Magnitudes which are proportional to the two first Magnitudes.



Let A be to B as C is to D and let E be a common measure of the magnitudes A, B it is required to find a like common measure of two ~~other~~ magnitudes remaining proportionate C, D.

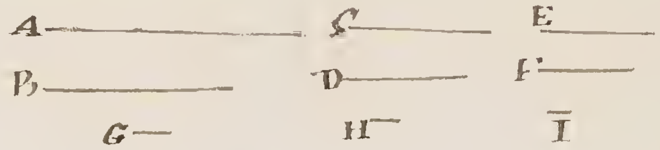
Since the Magnitude E exactly measures the magnitude A it will be known how many times the magnitude E is contain'd in A by taking the lesser E from the greater A continually till A be exhausted (3.1.). Then Divide the Magnitude C into ~~as many~~ the same number of parts ~~that~~ as A contains of E (2. pos.) and let F be one of them: Then is C the same multiple of F that A is of E and let the magnitude G be assumed which is the same multiple of F that B is of E by repeating F so many times as E is contained in B. Then since A, B are respectively the same multiples of the common measure E that C, G are of their common measure F therefore A will be to B as C is to G (Def. 6) But A is to B as C is to D (Hyp.) therefore the magnitude G is equal to the magnitude D. But G is the same multiple of F that B is of E therefore also D is the same

is the same Multiple of F that B is of E and so F is the same common measure of the Magnitudes C, D that E is of the Magnitudes A, B.

Prop. 2.

If two Magnitudes and again two others be proportional to two and the same magnitudes they shall be proportional to one another by equality.

Let A be to B as C to D and C to D as E to F then I say that A is to B as E is to F.

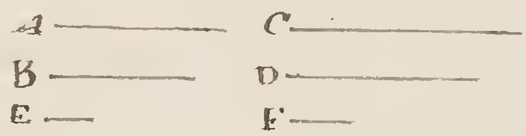


For assume G a common measure of A & B (Post. 3) and find H the like common measure of C & D that G is of A & B also I the like common measure of E & F that H is of C & D (1.) Then since A is the same Multiple of G that C is of H and whatever multiple C is of H the same is E of I therefore A is the same multiple of G that E is of I. Again in the same manner it is shewn that B is the same multiple of G as F is of I: therefore A, B are respectively the same multiples of a common measure G that E, F are of their common measure I and consequently A has the same proportion to B that E has to F (Def. 6.)

Prop. 3.

If four Magnitudes be proportional they will be so by ~~composition~~ alternately.

Let A be to B as C to D I say A is to C as B to D.

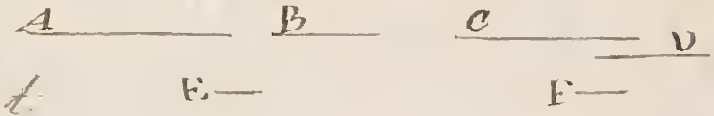


For E being assumed a common measure of A, B let F be found

the like common measure of C, D (1.)
Then since A is a multiple of E and C an equimultiple of F, therefore A is to C as E is to F (Ax.3) Again since E is an aliquot part of B and F the same aliquot part of D therefore E will be to F as B is to D (Ax3). But it has been shewn that A is to B as E is to F, therefore by equality of proportion A is to C as B to D.

Prop. 4.

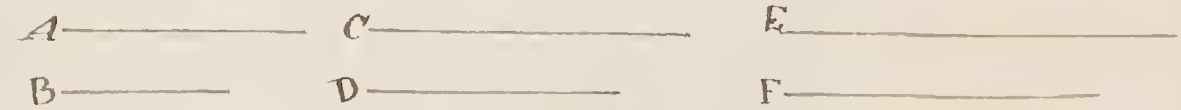
If four Magnitudes be proportional they will be so by Composition



Let A be to B as C is to D I say that A, C taken together are to B, D taken together as either antecedent A is to its consequent B.
For E, F being found as above since A is the same multiple of E that C is of F and B the same Multiple of E that D is of F (Def. 6.) therefore A, C taken together ~~are to~~ 2 will be the same Multiple of E, F taken together that A is of E and B, D together will be the same multiple of E, F that B is of E. Consequently A, C taken together are to B, D taken together as A is to B (Def. 6)
Again I say that antecedents & consequents compounded together are as the antecedents or as the consequents viz. A, B taken together are to C, D taken together as A to C or as B to D.
For since A is to B as C to D then by alternation (3.) A is to C as B to D and by Composition A, B together are to C, D together as A is to C or as B is to D by Equality.

Cor. If any number of Magnitudes be proportional two by two as any antecedent is to its Consequent so are all the antecedents to all the consequents.

Let A be to B as C to D as E to F.



I say as A is to B so are A, C, E together to B, D, F together.
For first by Composition A is to B as A, C together are to B, D, together; but A is to B as E is to F therefore by Equality E is to F as A, C together are to B, D together; and again by Composition E is to F as A, C, E together are to B, D, F together but E is to F as A is to B therefore by Equality A is to B as A, C, E together are to B, D, F together.

Prop. 5.

If four magnitudes be proportional they shall be so Dividedly.

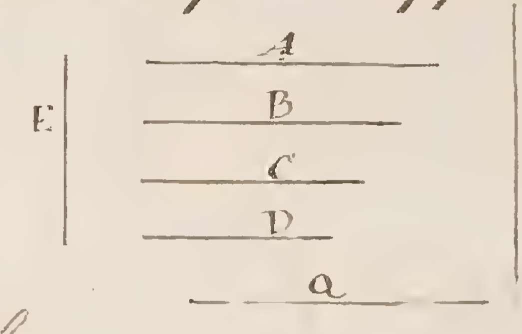
Let A be to B as C to D. I say first As A is to B so is the Excess of A above C to the excess of B above D.

For E, F being found as in the preceding then since A is the same multiple of E that C is of F the excess of A above C will also be the same multiple of the Excess of E above F. Again since B is the same multiple of E that D is of F, the excess by which B exceeds D will also be the same multiple of the excess by which E exceeds F (Ax.). Therefore the excess of A above C and of B above D are respectively the same multiples of the excess of E above F that the magnitudes A, B are of E and so A is to B as the excess of A above C is to the excess of B above D (Def. 6).

Again I say A is to C or B to D as the excess of A above B is to the excess of C above D.
For since A is to B as C to D by Alternation A is to C as B to D and by Division A is to C or B is to D as the excess of A above B is to the excess of C above D.

Prop. 6.

In a Perturbate Ratio the extremes are reciprocally proportional.



If A be to B as C to D and E be to B as C to F which is a perturbate Ratio then I say that the extremes A, D are reciprocally proportional to the extremes E, F viz. A is to E as F is to D.
For let G be assumed in the same proportion to F as A is to B or as C is to D (C.1.). Then since A is to B as G is to F and E is to B as C to F which is an ordinate Ratio therefore by Equality A is to G as E is to C (   ) But G is to F as C to D therefore also by Equality A is to E as F is to D (   ).
Cor. When in two Ranks of Proportionals the Means or the extremes in both or the Means in one and the Extremes in the other are the same or equal Magnitudes of the same kind then the other terms in the one order are reciprocally proportional to the other terms in the other.
As if A be to B as C to D and B be to E as F to C or if B be to A as D to C and B be to E as F to C then by Alternation it is reduced to a Perturbate Ratio and therefore universally A is to E as F to D.

Prop I.

Triangles & Parallelograms having the same altitude are to each other as their Bases.—

Let the triangles ABC, ABD and the parallelograms BE, BF have the same altitude or stand between the same parallels DC, EF. I say as the triangle ABC is to the triangle ABD or as the parallelogram BE is to the parallelogram BF so is the Base BC to the Base BD.

Assume I a common measure of the bases BC, BD and in BC take BG and in BD take DH each equal to I also join AG, AH. Then because BG repeated a certain number of times would exactly measure the Base BC, therefore right lines drawn to each extremity of these equal parts in BC would divide the triangle ABC into as many triangles as the base BC into parts and each of these triangles are equal to the triangle ABG because standing upon equal bases and between the same parallels (19.1.) therefore the triangle ABC is the same multiple of the triangle ABG as the Base BC is of the parts BG or its equal I. For the same reason is the triangle ABD the same multiple of the triangle ADH which the base BD is of I. But the triangles ABG. ADH are equal between themselves because standing upon the equal bases BG, DH and between the same parallels DC EF (19.1.). Therefore the triangles ABC, ABD are respectively the same multiples of the common magnitude ABG which the Bases BC BD are respectively of the common

Magnitude S. Consequently the triangle ABC is to the triangle ABD as the Base BC is to the Base BD (C.D.5.).
In the very same manner is it demonstrated of parallelograms only substituting the word parallelogram for triangle.

P. 4th

Triangles & Parallelograms which have one angle in the one equal to one angle in the other or the two angles together equal to two right angles are to each other as the Rectangles under the Sides about the said angles.
Let ABC, DEC be two triangles in which the angle ACB of the one is equal to the angle DCE of the other or the angles ACB, DCE together equal to two right angles.
I say the triangle ABC is to the triangle ~~DCE~~ DEC as the rectangle ACB to the rectangle DCE.
Join BD. Then the triangle ABC is to the triangle DBC as AC is to DC (1.). But AC is to DC as the Rectangle ACB to the rectangle DCB (Ax. 5.); therefore the triangle ABC is to the triangle DBC as the rectangle ACB to the rectangle DCB.
For the same reason is the triangle DEC to the triangle DBC as the rectangle DCE to the rectangle DCB. Therefore by equality of Ratio ( ) the triangle ABC is to the rectangle as the triangle DEC to the rectangle DCE and by Alternation the Triangle ABC is to the triangle DEC as the rectangle ACB to the rectangle DCE.
In the very same manner it is demonstrated of parallelograms only substituting parallelogram for triangle.
Cor. Triangles which are equiangular are as the rectangles under their homologous sides.

Triangles & Parallelograms which stand upon the same or equal Bases are as their Altitudes.

Let ABC, DBC be two triangles standing upon the same BC. I say the triangle ABC is to the triangle DBC as the altitude AE to the altitude DF.

Thro' A, D draw the Right lines AG, DH parallel to BC (8.1) and at C draw CG perpendicular to BC meeting AG in G and DH in H (13.1.) also join BG, BH. Then since AG, DH, BC are parallel and AE, GC, HC, DF are each perpendicular to BC, therefore AE will be equal to GC and DF to HC (1. C. 17. 1.). But since BC is perpendicular to GC, BC will be the common altitude of the triangles GBC, HBC and the triangle GBC will be to the triangle HBC as GC to HC (1.) i.e. as AE to DF because it has been shewn that AE is equal to GC and DF equal to HC. But also the triangle ABC is equal to the triangle GBC and the Triangle DBC equal to the triangle HBC (19. 1.) and the triangle GBC is to the triangle HBC as AE to DF therefore (by Ax. 5.) the triangle ABC is to the triangle DBC as the Altitude AE to the Altitude DF.

And if they had stood upon different but equal bases it will be the same because the equal Bases may be applied to and will coincide with each other.

And in the very same manner it is demonstrated of parallelograms only substituting parallelogram for triangle.

If ~~from~~ four Right lines be proportional, the rectangle contained under the extremes is equal to the Rectangle under the means, and if the Rectangle contained under the extremes be equal to the rectangle contained under the means then are the four Right lines proportional.

Let AB, BC, BD, BE be four right lines in proportion so that AB be to BC as BD to BE. I say the rectangle contained under the extremes AB, BE is equal to the rectangle contained under the means BC, BD.

Let AB, BC be placed in one straight line and BD, BE in another but so as AB, BD and consequently BC, BE be at right angles to each other and compleat the rectangles AE, EC, CD.

Then the rectangle AE, EC have the same Altitude EB and therefore AE is to EC as the Base AB to the Base BC (1.) But AB is to BC as BD to BE. Again since the rectangles DC, CE have the same altitude BC the Rectangle DC is to the Rectangle EC as BD is to BE. But the rectangle ~~under~~ AE is also to the rectangle EC as BD to BE therefore by equality of proportion the Rectangle AE is to the Rectangle EC as the rectangle DC is to the rectangle EC and alternately the rectangle AE is to the rectangle DC as EC is to EC that is the rectangles AE, DC are proportional to one and the same magnitude and therefore are equal between themselves (Ax. 5.) but the Rectangle AE is that contained AB, BE and the rectangle DC is that contained under BD, BC therefore the rectangle under the extremes AB, BE is equal

to the rectangle under the means BD, BC.
And conversely, if the Rectangle AE contained under the extremes AB. BE be equal to the rectangle DC contained under the means BD. BC then I say that AB is to BC as BD is to BE.
For the rectangle AE is to the rectangle EC as AB to BC but AE is equal to the Rectangle DC (Hyp.) therefore DC is to EC as AB to BC but also DC is to EC as BD to BE (1.) therefore (by equality of Proportions) AB is to BC as BD to BE.
Cor. If three Right lines be proportional the rectangle under the extremes is equal to the square of the mean; and if the rectangle under the extremes be equal to the square of the mean then the three Right lines are proportional.
For when BC is equal to BD then AB is to BC as BC to BE and the Rectangle under BC, BD is the square of BC therefore the rectangle under the extremes AB, BE is equal to the Rectangle under the means BC, BD that is to the square of the mean BC.
And conversely since the Rectangle under AB, BE is equal to the square of BC that is to the rectangle under BC, BD therefore AB is to BC as BD to BE. But BC is equal to BD; therefore AB is to BC as BC to BE.

P. 5th

Equal triangles & equal parallelograms having one angle of the one equal to one angle of the other or the two angles together equal to two Right angles have their sides about the equal angles reciprocal and conversely if triangles & parallelograms which have one

angle in the one equal to one angle in the other or the two angles together equal to two right angles have the sides about the equal angles reciprocal they will be equal between themselves.

Let the Triangles ABC, DEC or the parallelograms CF, CG be equal between themselves and have the angles ACB, DCE one in each equal also or together equal to two Right angles. I say the sides about these angles will be reciprocal; that is AC will be to DC as EC to BC. For since the triangles & parallelograms have one angle in one equal to one angle in another or the two angles together equal to two Right angles they will be as the rectangles under the sides about these angles (4.). But the triangles or parallelograms are equal between themselves (Hyp.) therefore also the rectangles under the sides are equal between themselves (Ax. 5.); and so the rectangle under AC, CB is equal to the rectangle under DC, CE. But if the rectangle under the extremes be equal to the rectangle under the means the right lines are proportional viz AC is to DC as EC is to BC.

And conversely if the triangles ABC, DEC or the parallelograms CF, CG have one angle ACB of the one equal to one angle DCE of the other or the two angles together equal to two right angles and the sides about these angles reciprocal viz ~~AB be to BC~~ AC be to DC as EC is to BC then I say the triangles or the parallelograms will be equal between themselves.

For since AC is to DC as CE to CB the rectangle under AC,

CB will be equal to the rectangle under DC. CE. But the triangles ABC, DEC or the parallelograms CF, CG having the angle ACB of the one equal to the angle DCE of the other or the angles ACB, DCE together equal to two right angles will be as the rectangles under the sides AC, CB and DC. CE (4.) But these rectangles have been shewn to be equal therefore also the triangles themselves or the parallelograms will be equal between themselves.

P. 6.

Triangles which have two angles in each severally equal between themselves or only one in each equal to one in the other but the other two taken together equal to two Rights angles then shall the sides subtending these angles in each taken Correspondently be proportional.

Let ABC, ADE be two triangles having two angles in each severally equal viz the angle BAC equal to the angle DAE and the angle ABC equal to the angle ADE; Or if the angles at A be equal but the angles ABC, ADE together equal to two right angles I say BC is to AC as DE to AE.

For since the angles ABC ADE are either equal between themselves or together equal to two right angles therefore the triangle ABC is to the triangle ADE as the rectangle under AB, BC is to the Rectangle under AD, DE (4.). Again since the angles BAC, DAE are ~~either~~ equal between themselves the triangle ABC is to the triangle ADE as the rectangle under BA, AC is to the rectangle under DA, AE (4.). Therefore by equality of proportion the rectangle under AB, BC is to the rectangle under AD, DE as the rectangle

under BA, AC is to the rectangle under DA, AE that is (alternately) the rectangle under AB, BC is to the rectangle under BA, AC as the rectangle under AD, DE is to the rectangle under DA, AE. But the rectangle under AB, BC is to the rectangle under BA, AC as BC is to AC (C.1.) therefore BC is to AC as the rectangle under AD, DE to the rectangle under DA, AE. But also DE is to AE as the rectangle under AD, DE is to the rectangle DA, AE (C.1.) therefore again by Equality BC is to AC as DE is to AE.
Cor. Equiangular triangles have their homologous sides proportional.

P. 7.

Equiangular triangles are in the Duplicate ratio of their homologous sides.
Let ABC, ADE be two Equiangular Triangles. I say they shall be in the Duplicate Ratio of any two homologous sides AB, AD that is the triangle ABC shall be to the triangle ADE as the square of AB is to the square of AD.
~~Let~~ Of the remaining angles let BAC be equal to DAE and ABC equal to ADE. Then the homologous sides being proportional AB is to BC as AD is to DE (C. 6) ~~therefore by equality of proportion~~. But the square of AD is to the rectangle under AD, DE as AD is to DE (C.1.) therefore by equality of proportion AB is to BC as the square of AD to the Rectangle under AD, DE. but again AB is to BC as the square of AB is to the rectangle under AB. BC (C.1.) Therefore also by equality of proportion the square of AB is to the rectangle under AB, BC as the square of AD is to the rectangle under AD, DE and alternately the square of AB is to the square of AD

as the Rectangle under AB, BC is to the Rectangle under AD, DE. But since the angles ABC, ADE are equal between themselves the triangle ABC is to the triangle ADE as the rectangle under AB, BC is to the rectangle under AD, DE (C.4.) therefore by equality of proportion the triangle ABC is to the triangle ADE as the square of AB is to the Square of AD.

P. 8.

If there be two triangles whose sides are proportional the angles subtended by the homologous sides shall be equal between themselves are the triangles be equiangular. Let ABC, DEF be two triangles whose sides are propor-tional that is AB is to BC as ~~AD~~ DE is to EF and AB is to AC as ~~as~~ DE is to DF. I say the triangles are equiangular and the angles subtended by the homologous sides are equal viz the angle BAC is equal to the angle EDF the angle ABC equal to the angle DEF and the angle ACB equal to the angle DFE.

In AB (produced if necessary) take AG equal to DE and in AC take AH equal to DF and join GH. Then since AB is to AC as DE to DF and AG is equal to DE and AH to DF there-fore AB will be to AC as AG to AH but the angles included by these homologous sides are equal being the common angle at A, therefore the triangles ABC, AGH are equian-gular (9.) viz the angles ABC, AGH subtended by the homologous sides ~~AB, AG~~ AC, AH and the angles ACB AHG subtended by the homologous sides AB, AG are equal between themselves. But the triangles ABC, AGH being equiangular the homologous sides are proportional

(1. C. 6.) that is AB is to BC as AG to GH, & also 20
AB is to AC as DE to EF, therefore ( ) AG is to GH
as DE to EF, but AG is equal to DE, therefore also GH
will be equal to EF ( ). The triangles AGH, DEF have
therefore the three sides of the one equal to the three
sides of the other, each to each, and consequently the
angles subtended by the equal sides will be equal (6.1)
viz, the angle AGH will be equal to the angle DEF,
the angle AHG to the angle DFE, and the angle GAH to
the angle EDF; but the angles AGH, AHG, GAH are seve-
-rally equal to the angles ABC, ACB, BAC, therefore
the angle DEF will be equal to the angle ABC, the
angle DFE to the angle ACB, & the angle EDF to the
angle BAC.

Otherwise.

Take AG equal to DE, & draw GH parallel to BC (8.1)
cutting AC in H. Then the angle ABC is equal to the
angle AGH; the angle ACB equal to the angle AHG
(9.1.) and the angle at A being common to both the
triangles ABC, AGH, they are equiangular and the
homologous sides proportional (1. C. 6). Therefore AB
is to BC as AG to GH, but AB is to BC as DE to EF,
therefore, by equality of proportion, AG is to GH
as DE to EF. But AG is equal to DE, therefore also GH
is equal to EF ( ). And in the same manner is it
shewn that AH is equal to DF. Therefore the three
sides of the triangle AGH are equal to the three sides
of the triangle DEF, each to each, & consequently
the angles in each, subtended by equal sides, will
be equal, viz, the angle AGH equal to the angle DEF,

the angle AHG equal to the angle DFE, & the angle GAH equal to the angle EDF. But the angle ABC is equal to the angle AGH, therefore the angle ABC will be equal to ye angle DEF; and for the same reason is the angle ACB equal to the angle DFE, and the angle BAC equal to the angle EDF. Therefore the triangles ABC, DEF are equiangular, & the angles in each subtended by homologous sides, equal between themselves.

P. 9.

If two triangles have one angle of the one equal to one angle of the other, & the sides about these equal angles be proportional; then the triangles are equiangular, & the angles subtended by equal sides, are equal.

Let ABC, DEF be two triangles, having one angle BAC of the one equal to one angle EDF of the other, & the sides abt these equal angles proportional, viz, AB to AC as DE to DF, I say the triangles are equiangular, & the angles subtended by homologous sides, viz, ABC, DEF, and ACB, DFE, are equal between themselves. —

(In AB, produced if need be, take AG equal to DE & draw GH parallel to BC (8. 1.) The angle ABC is equal to the angle AGH, the angle ACB equal to the angle AHG (9. 1.) & the angle at A being common, the triangles ABC, AGH are equiangular, & the homologous sides are proportional; viz, AB is to AC, as AG is to AH (1. 6. 6) that is, because AG is equal to DE, AB is to AC as DE is to AH; but AB is to AC as DE to DF, therefore AH & DF having ~~having~~ one & the same proportion, are equal to each other ( ). The triangles AGH, DEF have therefore two sides AG, AH of the one equal to two sides

DE. DF of the other, & the contained angles GAH, EDF are equal (Hyp.) the triangles are therefore equiangular, & the angles subtended by equal sides, will be equal, viz, the angle AGH equal to the angle DEF, & the angle AHG equal to the angle DFE ( ). But the angle ABC is equal to the angle AGH, and the angle ACB equal to the angle AHG, therefore the angle ABC is equal to the angle ~~AGH, & the angle ACB equal to the angle AHG~~ DEF, & the angle ACB to the angle DFE.

P. 10.

A right line drawn parallel to one side of a triangle cuts the other sides proportionally; & if two sides of a triangle be cut proportionally by a right line drawn, that right line will be parallel to the other side.

Let ABC be a triangle, & parallel to one of its sides BC let a right line DE be drawn, I say, DE will cut the other sides AB, AC proportionally in D, E.

For, since DE is parallel to BC, AB will be to AC as AD to AE, and (alternately) AB is to AD as AC is to AE, also (dividedly) BD is to AD as CE is to AE.

And conversely, if DE be drawn, so that BD be to AD as CE to AE, then I say, DE is parallel to the other side BC. For, since BD is to AD as CE to AE, then (compoundedly) AB is to AD as AC is to AE, & (alternately) AB is to AC as AD is to AE. Therefore the triangles ABC, ADE have one angle common at A, & the sides including this common angle are proportional, consequently the angles subtended by homologous sides are equal (9.)

viz the angle ABC is equal to the angle ADE. But when a right line falls upon two right lines making the internal angle equal to the external and opposite angle on the same side, these two right lines shall be parallel ( ). DE is therefore parallel to BC.

Prop. 11.

If at any angle of a triangle, a right line be drawn to make equal angles with the sides containing it, whether internally or externally, and cut the Base; the segments of the Base intercepted between its extremities & the point of section, shall be as the adjacent sides of the triangle. And, if from the Vertex of a triangle a right line be drawn to cut the base in the proportion of the sides, that right line shall make equal angles with the sides, ~~ex~~ internally or externally, as the point of section is within or without the Base.

Let the right line CD be drawn from the Vertex C of the triangle ABC making with the sides CA, CB the equal angles ACD, BCD or externally the equal angles ACE, BCD. I say the base AB is cut thereby in the proportion of the sides viz AD is to BD as AC to BC.

For when the right line CD is drawn internally the triangles ACD, BCD have one ACD equal to one angle BCD of the other and another angle ADC of the one taken with another angle BDC of the other is equal to two right angles. Also when CD is drawn externally since the angle ACE is equal to the angle BCD the angles ACD, ACE together

will be equal to the angles ACD, BCD together; but the angles ACD, ACE together are equal to two right angles therefore the angles ACD, BCD together are equal to two right angles. And so whether the Right line CD be drawn internally or externally, the triangles have one angle in each equal or the same and another angle of the one taken with another angle of the other equal to two right ~~ones~~ angles therefore the sides subtending these angles are proportional (6.) viz AD is to AC as BD to BC and alternately AD is to BD as AC to BC.

And conversely if from the Vertex C of the triangle ABC a right line be drawn to cut the base AB in the proportion of the sides viz that AB be to BD as AC to BC then CD shall make equal angles with the sides AC, BC internally or externally as CD is drawn ~~without~~ within or without the triangle ABC.

For since AD is to BD as AC to BC (by alternation) AD is to AC as BD to BC. Therefore when CD is drawn within the triangle ABC the two triangles ACD, BCD have two sides of the one proportional to two sides of the other and the angles ADC, BDC subtended by the homologous sides AC, BC are together equal to two right angles therefore the remaining angles ACD, BCD subtended by the other homologous sides ~~AC, BC~~ AD, BD are equal between themselves and so the right line CD when drawn within the triangle ABC makes equal angles internally with the sides AC, BC.

Again when CD is drawn without the triangle ABC then

the triangles ACD, BCD have two sides AD, AC of the one proportional to two sides BD, BC of the other and the angles ADC, BDC subtended by the homologous sides AC, BC are ~~equal between themselves and so the right line CD together equal to two right angles~~ equal between themselves being one and the same angle therefore the remaining angles subtended by the other homologous sides AD BD viz ACD, BCD must be either equal between themselves or together equal to two right angles ( ). But they cannot be equal because the one is only a part of the other; ACD, BCD are therefore equal to two right angles. But the angles ACD, ACE are equal to two right angles, therefore the angles ACD, BCD are equal to the angles ACD ACE {and taking away from both the common angle ACD the remaining angle BCD is equal to the remaining angle ACE. And so the right line CD, when drawn without the triangle ABC makes equal angles externally with the sides AC, BC.

P. 12.

If two triangles have two sides of the one proportional to two sides of the other and of the correspondent angles two be equal between themselves the remaining two if of like affection shall also be equal between themselves and the triangles equiangular but if of different affection they shall together be equal to two right angles.

Let ABC, DFE be two triangles having two sides of the one proportional to two of the other viz AB to AC as DF to DE and of the correspondent angles subtended by homologous sides if two

if two ABC, DFE be equal between themselves the other two ACB, DEF if of like affection shall also be equal between themselves and the triangles be equiangular but if of different affection they shall together be equal to two right angles.

In AB, AC produced if necessary take AG equal to DF and AH equal to DE and join GH. Then since AB is to AC as DF is to DE and DF is equal to AG and DE equal to AH it will be as AC is to AB so is AG to AH and alternately & dividedly BG is to AG as CH is to AH therefore GH is parallel to BC (10.6.) and consequently the angle AGH is equal to the angle ABC and the angle AHG equal to the angle ACB ( ). But the angle DFE is equal to the angle ABC therefore the angle AGH is equal to the angle DFE.

Then if the angles ACB, DFE be of like or different affection the angles AHG, DFE will also be of like or different affection because the angle AHG is equal to the angle ACB. Therefore the triangles AGH, DFE having two sides AG, AH of the one equal to two sides DF, DE of the other each to each and of the correspondent angles subtended by equal sides the two AGH, DFE are equal between themselves therefore the remaining two AHG, DFE if of like affection also equal between themselves ( ). But the angle AHG is equal to the angle ACB therefore the angle ACB is equal to the angle DEF and the triangles ABC, DFE equiangular having two angles in each equal.

But if the angles AHG, DFE be of different affection they will be together equal to two right angles ( ).

But the angles AHG, DEF are equal to the angles ACB, DEF therefore also the angles ACB, DEF are together equal to two right angles.

Cor. If two triangles have two sides of the one proportional to two sides of the other and if two of the correspondent angles be equal between themselves but the other two are unequal then the unequal angles must be of different affection and together equal to two right angles.

For they either are of like or different affection; if they be of like affection they will be equal between themselves which is impossible therefore they must be of different affection and consequently together equal to two right angles.

Cor. 2. If two triangles have two sides of the one proportional to two sides of the other and if two of the correspondent angles be equal between themselves; but the other two together ~~equal to~~ less than two right angles then these two shall be of like affection and consequently equal between themselves and the triangles.

For they are either of like or of different affection if of different affection then they will together be equal to two Right angles which is impossible therefore they are of like affection and consequently equal between themselves and the triangles equiangular.

P. 13.

If two triangles have two sides of the one proportional to two sides of the other and two of the correspondent angles be together equal to two right angles the other two shall be equal between themselves.

If the triangles ABC, ADE have two sides AC, CB proportional to two sides AE, ED, and if two of the correspondent angles BAC, DAE subtended by homologous sides CB, ED be together equal to two right angles the remaining two ABC, ADE subtended by homologous sides AC, AE shall be equal between themselves.

For since AC, AE are supposed to coincide in one straight line AB, AD will also coincide in one right line because the adjacent angles made with AC viz BAC, CAD are together equal to two right angles. Draw EF parallel to CB then the triangles ABC, AFE are equiangular (   ) and AC is to CB as AE is to EF (   ). But AC is to CB as AE to ED therefore EF is equal to ED (   ) and the angle EDF equal to the angle EFD (   ). But the angle EFD is equal to the angle ABC (   ) therefore the angle EDF or EDA is equal to the angle ABC.

P

If thro' the vertex of any Angle a right line be drawn making equal angles with the sides whether internally or externally and from any two points assumed in the sides two right lines be drawn parallel to each other and fall on the former line drawn these two parallels shall be in the proportion of the segments assumed of the sides.

Let ACB be a right lined angle and thro' the vertex C let a right line DE be drawn to make equal angles with the sides AC, BC whether internally or externally; also from any two points F, H in the sides AC, BC let two right lines FG, HI be any how drawn but parallel to each other and fall upon DE in G, I: I say FG is to HI as FC to CH.

Since DE makes equal angles with the sides AC, BC (Vid. n.1.) the angle ACE will be equal to the angle BCE that is equal to the angle DCH ( ) and since DE falls upon the parallels FG, HI the external angle FGC taken with the internal alternate angle HIC will be equal to two right angles. ( ). But (v. n. 2.) since DE makes equal angles with the sides AC. BC the angle ACD will be equal to the angle BCD and the angles ACD. ACE together equal to the angles ACE BCE together. But the angles ACD. ACE together are equal to two right angles therefore also the angles ACE BCE or what is the same thing ACE, DCH together are equal to two right angles and the right line DE falling upon the parallels FG, HI the internal angle FGC is equal to the external angle HIC on the same side ( ) therefore whether DE be drawn within or without the angle ACB and in every position which the parallel right lines FG HI can have, the triangles ACG, HCI have two angles FCG HCI equal to each other and two others FGC. HIC together equal to two right angles or the angles FCG, HCI are equal to two right angles and then the angles FGC, HIC are equal between themselves or the angles FCG, HCI may be equal between themselves and also the angles FGC, HIC consequently in every case the sides subtending these angles are proportional ( ) viz FG is to FC as HI to HC and by alternation FG is to HI as FC is to HC.

Scholium. Prop. 11 is only a particular case of this general proposition.

In a Right-angled triangle a perpendicular drawn from the right angle to meet the base resolves the whole into triangles similar to each other and also to the whole.
In the right-angled triangle ACB let the Right line CD be drawn from the right angle ACB perpendicular to the base AB. I say the triangles ADC, CDB are similar to each other and also to the whole ACB.
For the triangles ACB, ADC have one angle at A common to both and another angle ACB is equal to another ADC of the other being each of them a right angle therefore the remaining angles ABC, ACD are equal between them selves ( ) and the triangles ACB, ADC are equiangular. In the very same manner it may be proved that the angle BAC is equal to the angle BCD and of consequence the triangles ACB, CDB also equiangular. Thus the triangles ADC, CDB are each equiangular to the whole triangle ACB and therefore are equiangular to each other. But equiangular triangles are also similar to each other ( ) There-fore the triangles ADC, CDB are similar to each other and each similar to the whole triangle ACB.
Cor. 1. Either side of a right angled triangle is a mean proportion between the Hypothenuse and the adjacent segment of the Hypothenuse made by a perpendicular falling upon it from the right angle.
For the triangles ACB, ADC being similar their homologous sides are proportional viz AB is to AC as AC is to AD.
Cor. 2. The perpendicular let fall from the right angle

of a Right-angled triangle upon the Hypothenuse is a mean proportional between the segments of the Hypothenuse.

For the triangles ADC, CDB being similar their homologous sides are proportional viz AD is to DC as DC is to DB.

Cor. 3. The sides about the right angle are reciprocal to the Hypothenuse and perpendicular drawn from the right ~~line~~ angle upon the Hypothenuse.

For the triangles ADB, CDB being similar their homologous sides are proportional viz AB is to AC as CB is to CD.

P. 15.

Similar Polygons are resolved into similar triangles equal in number and homologous to each other and the Polygons are in the Duplicate Ratio of their homologous sides.

Let ABCDE, FGHKL be similar polygons in which the angles according to the order of the Letters are equal to each other and the sides in the same order homologous; I say they may be resolved into an equal Number of similar triangles homologous to the wholes and the polygons are in the duplicate Ratio of any two homologous sides.

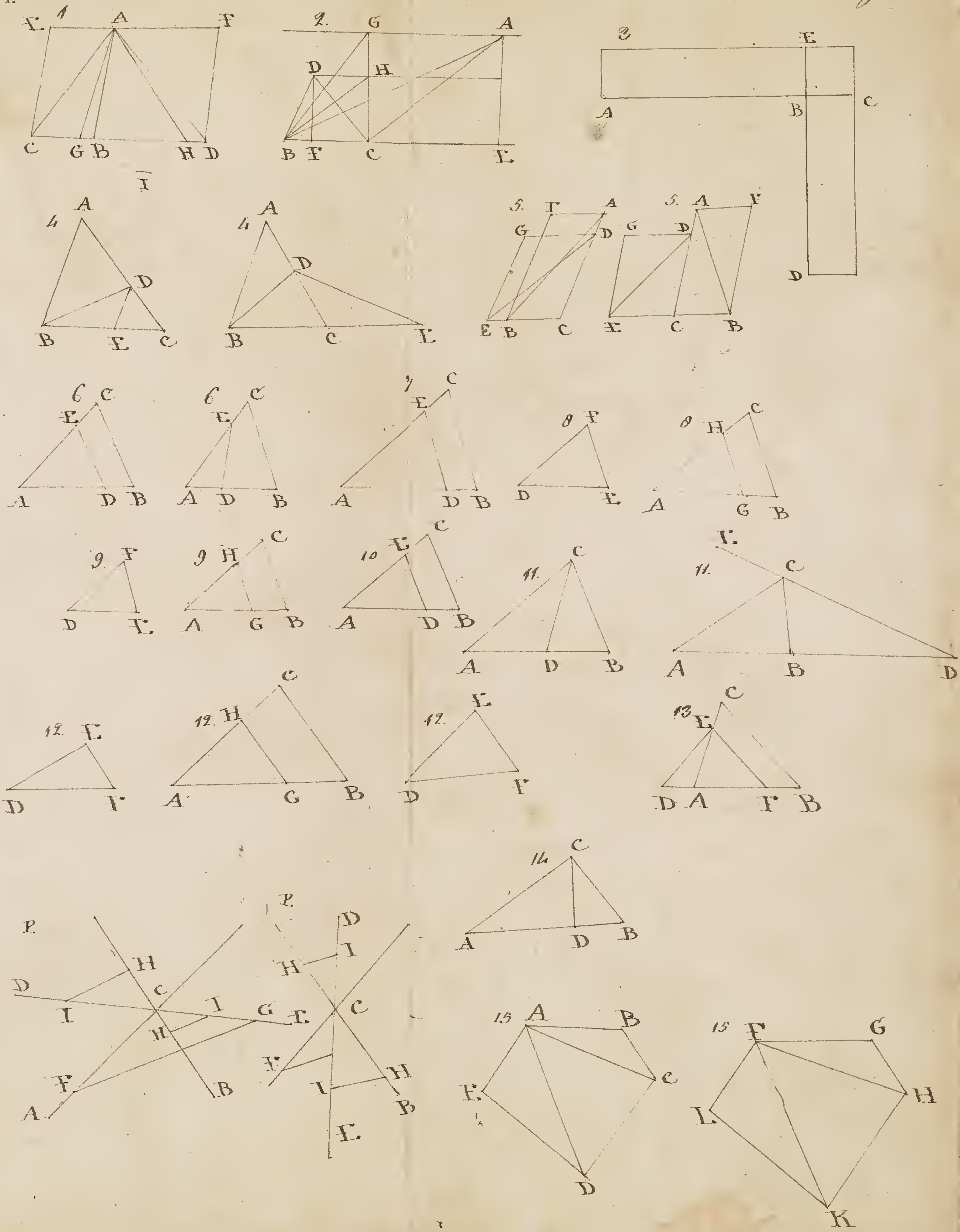
Draw the Diagonals AC, AD and also FH, FK.

Then since AB is to BC as FG is to GH and the angle ABC is equal to the angle FGH the triangles ABC, FGH will be equiangular (   ) and consequently similar (   ). For the same reason are the triangles AED, FLK similar to each other. Again since the angle BCD is equal to the angle GHK

and the angle ACB equal to the angle FHG therefore the remaining angles ACD. FHK are equal between themselves. For the same reason are the angles CAD, HFK equal between themselves therefore the triangles ACD. FHK having two angles in each mutually equal are equiangular ( ) and consequently similar. Thus the polygons are resolved into an equal number of similar triangles.

I say also that they are homologous to the wholes that is any two similar triangles are to each other as the whole polygons; and Polygon is to Polygon in the Duplicate ratio of any two homologous sides.

For the triangles ABC. FGH being equiangular are in the duplicate Ratio of the homologous sides AC, FH ( ) But the equiangular triangles ACD. FHK are also in the same duplicate Ratio of AC to FH therefore the triangles ABC. FGH are as the triangles ACD. FHK (by equality of proportion). For the very same reason are the triangles ACD. FHK as the triangles AED. FLK therefore again by equality of proportion the triangles ABC. FGH are as the triangles AED. FLK. But the antecedents are ABC. ACD. AED and the consequents FGD. FHK, FLK and the antecedents make up the whole Polygon ABCDE also do the consequents the whole Polygon FGHKL and Antecedent is to consequent as all the antecedents are to all the Consequents ( ) Therefore the triangle ABC is to the triangle FGH as the polygon ABCDE is to the polygon FGHKL. But the triangle ABC is to the triangle FGH in the duplicate ratio of AB to FG ( ) Therefore also (by equality of proportion)



the polygon ABCDE is to the polygon FGHKL in the duplicate Ratio of AB to FG.

Cor. Similar polygons are to each other in the Duplicate ratio of their Perimeters.

For the correspondent sides are proportional to each other two by two and the sides of one Polygon are antecedents to the correspondent sides of the other Polygon as consequents the whole Perimeter of the one is to the whole perimeter of the other as any one side of the one is to an homologous side of the other ( ) and the squares of the perimeters are as the squares of two homologous sides but the Polygons are as the squares of two homologous sides therefore the Polygons are as the squares of their Perimeters.

P. 16.

If four Right lines be proportional similar figures described upon the antecedents as homologous sides are proportional to similar Right-lined figures describ'd upon the consequents as homologous sides and if similar right-lined figures described upon two Right lines as homologous sides be proportional to similar right-lined figures described upon two other right lines as homologous sides these four Right lines shall be proportional.

Let the four right lines AB, CD, EF, GH be proportional and let two similar right-lined figures ABF. CDK be described upon AB, CD as homologous sides of these figures also in like manner on EF, GH any other two similar

right-lined figures EL, GO: I say as ABI is to CDK so is EL to GO.

For since AB is to CD as EF to GH the square of AB will be to the square of CD as the square of EF is to the square of GH ( ). But the figures ABI, CDK being similar ABI is to CDK as the square of AB to the square of CD. Therefore by equality of proportion ABI is to CDK as the square of EF to the square of GH. But also the figures EL, GO being similar are as the squares of their homologous sides ( ) viz EL is to GO as the square of EF to the square of GH. Therefore by equality of proportion ABI is to CDK as EL to GO.

And conversely if the similar right-lined figures describ'd AB, CD as homologous sides be proportional to the similar figures described upon EF, GH as homologous sides AB will be to CD as EF to GH.

For the similar figures ABI, CDK are as the squares of their homologous sides AB, CD. but the figures ABI CDK are as the figures EL, GO. Therefore the figures EL, GO are as the squares of AB, CD by Equality of Ratio. But the figures EL, GO being similar are as the squares of their homologous sides EF, GH, therefore (by equality of Ratio) the squares of AB, CD are as the squares of EF, GH and consequently AB, CD are proportional to EF, GH ( ).

P. 17.

If two Circles meet in one common point and through the point of Concourse two Right lines be drawn making

alternately equal angles with the Diameter drawn from the same point of Concourse and either both within or both without the angle made by the Diameter the segments of these right lines intercepted by the Circles shall be reciprocal.

Let ABF, ACE be two circles meeting in A and let the Diameters AB, AC be drawn also through the same point A, two right lines AD, AF, either both within or both without the angle BAC cutting the Circles in D, E and F, G and making equal angles alternately with the Diameters viz the angle BAG equal to the angle CAE. I say the segments AD, AE and AF, AG shall be reciprocal viz AD shall be to AG as AF to AE.

Draw DG, FE and BG, CE. Then since the angle AGB as also the angle AEC is a right one because being each in a semicircle the angles BAG, ABG as also the angles CAE, ACE will together be equal to two right angles and therefore equal between themselves. But the angles BAG, CAG are equal between themselves (Hyp.) therefore the remaining angles ABG ACE are equal between themselves. But the angle ADG is equal to the angle ABG and the angle AFE equal to the angle ACE ( ) therefore the angles ADG, AFE are equal to each other as are also the angles DAG, EAF being either one and the same angle or vertical angles. Thus the triangles ADG, AFE are equiangular and the homologous sides proportional ( ) viz AD is to AG as AF is to AE.

Cor. 1. Hence the Rectangle under the Segments of the one Line drawn is equal to the Rectangle under the segments of the other.

For since AD is to AG as AF to AE the rectangle under

AD. AE will be equal to the Rectangle under AG, AF.

Cor. 2. If from the point of Concourse of two Circles their diameters be drawn and cuts both the Circles their segments shall be reciprocal and the Rectangles under their segments equal. For when AD coincides with AB, AG at the same time coincides with AC and therefore AB is to AC as AI to AH and the rectangle BAH equal to the rectangle CAI.

P. 18.

Two points being given in a right line to find a third point also therein such that the Distances thereof from the two former may be in a given Ratio.

Let AB be any given right line in which are given two points A, B it is required to find a third point D therein such that AD, BD may be in the given Ratio of 2 to p. If 2, p together be not greater than AB, assume any two right lines in the given Ratio of 2 to p which together may be greater than AB (as by doubling trebling &c the terms of the given Ratio) and with these right lines constitute the triangle ACB on the base AB ( ). At the vertex C draw CD either internally or externally making equal angles with the sides AC, BC and cutting the Base AB in D. D shall be the point required and AD shall be to BD as 2 to p.

For because CD makes equal angles with the sides AC, BC it will be as AD is to BD so is AC to BC ( ) But AC is to BC as 2 to p (Hyp.) therefore by equality of proportion AD is to BD as 2 to p.

Cor. 1 It is manifest that there are two points D, d which

answer the Conditions of the problem as the Line CD or Cd is drawn within or without the triangle the former of which is within the terms A. B but the latter without them.

Cor. 2. The Distances of one of the points thus found from the given terms of the Right line given are proportional to the correspondent distances of the other point from the same terms. Viz AD is to DB as Ad is to Bd.

P. 19th.

If two points be assumed in the Diameter of a Circle such that their distances from one extreme of the Diameter be proportional to their correspondent distances from the other; I say any two Right lines drawn from these points to meet in the Circumference of the Circle are in the same proportion.

In the Circle ABF let a Diameter AB be drawn in which let two points D, E be assumed such that AE may be to AD as BE to BD ( ) and to any point F in the Circumference let two Right lines EF, DF be drawn; I say EF is to DF in the same constant Ratio of AE to AD or of BE to BD.

Draw CF and take towards opposite parts CG equal to CD then since AC is equal to BC and the part CD equal to the part CG the remainder AD will be equal to the remainder BG. But EA is to AD as EB to BD that is dividedly EA is to AD as AB is to the excess of BD above AD. But since BG is equal to AD this excess will be equal to DG therefore EA is to AD as AB is to DG. But AB is double to AC and DG double to DC therefore their Halves AC, DC are in the same proportion viz EA is to AD as AC is to CD and by alternation and

and Composition EC is to AC as AC is to CD. But AC is equal to CF therefore EC is to CF as FC is to CD. Thus the triangles ACF, FCD have one angle at C common and the sides about this common angle are proportional therefore the triangles are equiangular and their homologous sides proportional ( ) Viz, FC is to EF as CD is to DF. ~~But it has been shewn~~ (and by alternation and because CF is equal to AC) as AC is to CD so is EF to DF. But it has been shewn that EA is to AD as AC to CD therefore by equality of proportion EA is to AD as EF is to FD.

Cor. If from either extremity of the Diameter a Right line be drawn to the point assumed in the Circumference it will make equal angles with the Right lines meeting in that point. Viz. if AF or BF be drawn either of them will make equal angles with EF, DF. For since EF is to DF as EA is to AD or as EB is to BD therefore AF internally and BF externally makes equal angles with AF, DF ( ).

P. 20

If in the Right Line passing thro' the Centre of two Circles a point be assumed whose distances from the Centres are in the Ratio of their semidiameters: a Right line drawn any how thro' that point to fall on the Circumferences in like parts will have its segments in the same Ratio of their semidiameters.

Let BOP, DCQ be two Circles O, C their centres and in the right line drawn thro' O, C let a point A be assumed such that AO may be to AC as OP to CQ. I say a right line drawn thro A to fall in like parts on the Circumferences in B, D

will have the segments AB, AD in the same Ratio of Cp to CQ. Draw OB CD then since the Right line ABD falls upon the Circumferences in like parts the angles ABO, ADC will be of like affection: But since BO is equal to Op and CD equal to CQ it will be as AO is to AC so is OB to CD and alternately as AO to BO so AC to CD. Therefore the triangles AOB, ACD have two sides of the one proportional to two sides of the other and of the correspondent angles viz subtended by homologous sides two BAO, DAC are equal between themselves being either one and the same or vertical angles while the remaining two ABO, ADC are of like affection; consequently the triangles are equiangular and the homologous sides are proportional ( ) viz AB is to AD as AO is to AC that is as Op is to CQ.

Schol: 1. When the point A falls within both Circles the angles made by the Right line drawn with the semidiameters viz the angles ABO, ADC will always be of like affection but then the parts of the circumferences where B, D fall are limited by the following Schol.

Schol: 2. When the point A falls without the terms O, C the Intersections B, D will be towards the same parts of A; when within towards different: Because otherwise the angles at A would neither be one and the same nor vertical angles but together equal to two right angles.

Cor. If in the common Diameter of two Circles a point be assumed such that the Distances thereof from the Centres are in the Ratio of the Semidiameters; I say a right line drawn any how thro' that point to cut both

Circumferences will cut off similar segments.
For let the right line ABD cut off the segments Dd, Bb and join also Ob, Cd. Then since the triangles ABO, ADC have been shewn to be equiangular and the sides AO, AC are homologous the angles ADC, ADC will be equal between themselves. But the angles BbO is equal to the angle ABO and the angle DdC equal to the angle ADC ( ). Therefore the angles ABO, BbO together will be equal to the angles ADC, DdC together and consequently the remaining angles BOb DCd will be equal between themselves ( ) and therefore the segments Bb, Dd are similar.

P. 22.

If in the common Diameter of two Circles a point be assumed such that the Distances thereof are in the Ratio of their semidiameters and if from this point right lines be any how drawn to cut the Circumferences in unlike parts. I say the Rectangles under their segments are equal between themselves.
Let BOP, DCQ be two Circles and in the right line drawn thro' O, C their Centers let a point A be assumed such that the Distances AO, AC may be as the semidiameters OP, CQ. I say if two right lines be drawn any how thro' A to cut the Circumferences in unlike parts as in D, b and H, g the segments of the right lines so drawn are reciprocal and their rectangles equal.
Let either of the right lines drawn cut the circumferences again b, d also let the common Diameter cut them in P p, Q, q and join CD, Cd, OB, Ob also QD, PB, pb.—
Then since the right line AD cuts of similar segments Bb, Dd

( ) the vertical angles BOb. DCd of the Isosceles triangles BOb DCd will be equal between themselves and consequently also the angles ABO, ADC at their Bases ( ) Therefore the right lines OB. CD are parallel ( ) and the angle AOB equal to the angle ACD ( ). But the angle AOB is double of the angle APB and the angle ACD double to the angle AQD therefore the angle APB is equal to the angle AQD. Again the angle Abp is equal to the angle APB ( ) therefore the angle Abp is equal to the angle AQD. Thus the triangles Abp, AQD have two angles Abp, AQD mutually equal and two other angles bAp. QAD also equal being either one and the same or vertical angles the triangles are therefore equiangular and their homologous sides are proportional viz Ab is to Ap as AQ is to AD. For the same reason is Aq to Ap as AQ is to AH. Therefore the means in both proportionals being the same the extremes are reciprocal that is Ab is to Aq as AH is to AD ( ) and consequently the rectangle under the extremes DA. Ab equal to the rectangle under the means HA. Aq.

Otherwise

Let BOP, DCQ be two Circles, O. C their Centres and in the right line drawn thro' O, C let a point be assumed such that the Distances AO. AC may be in the Ratio of the semi diameters OP, CQ; I say if two Right lines be drawn thro' A to cut the Circles in unlike parts as D, b and F, e the segments AD. Ab will be reciprocally as the segments AF. Ae and the rectangle under AD. Ab equal to the rectangle under AF. Ae.

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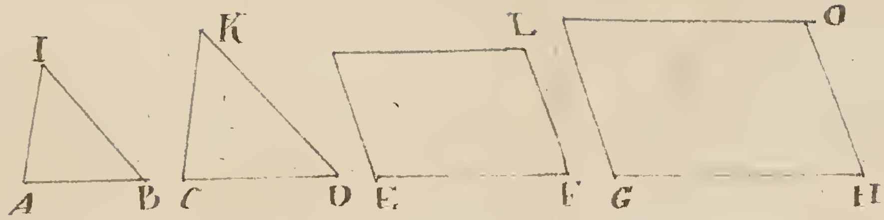

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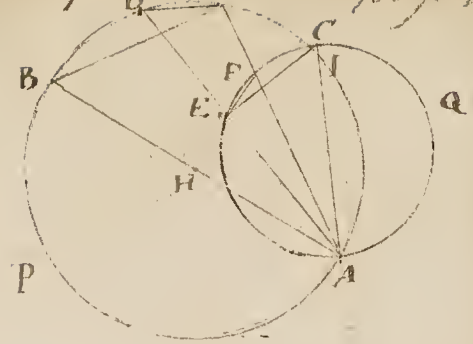

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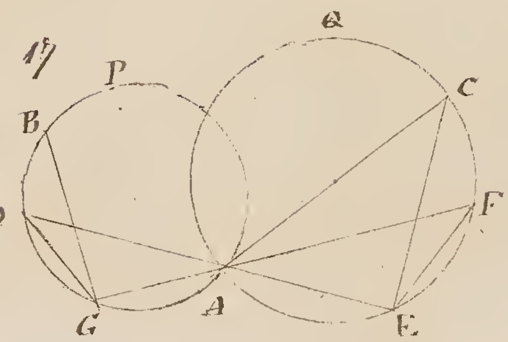

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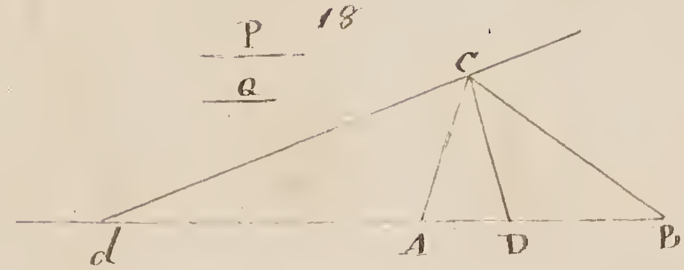

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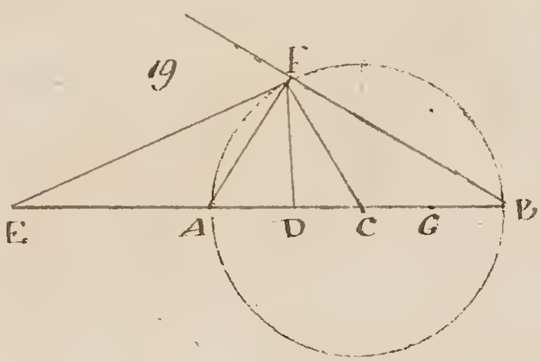

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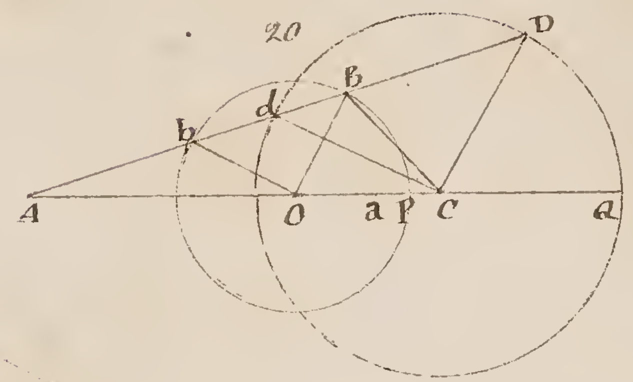

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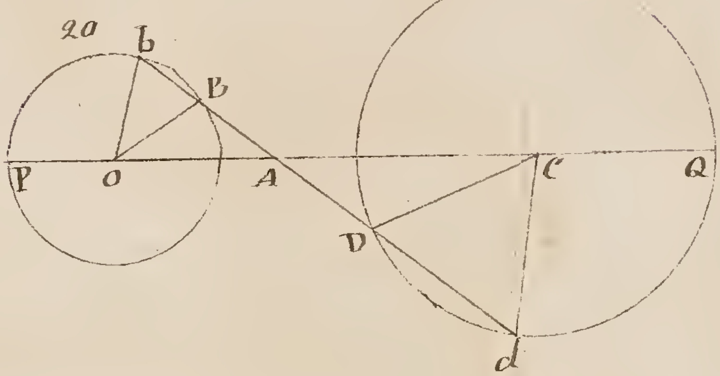

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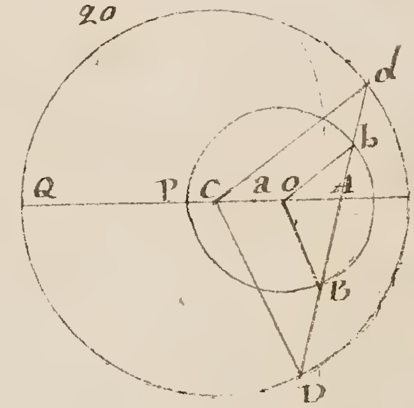

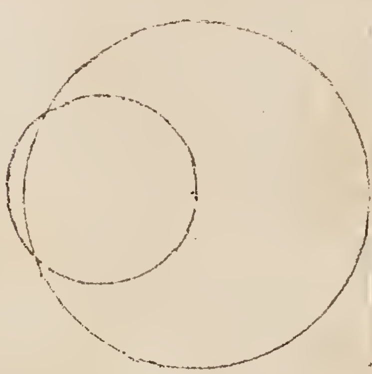

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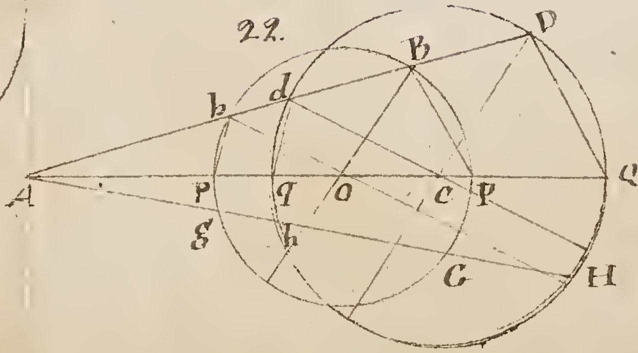

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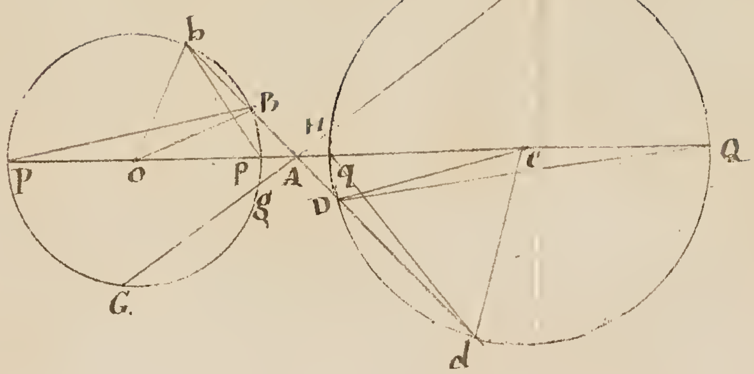
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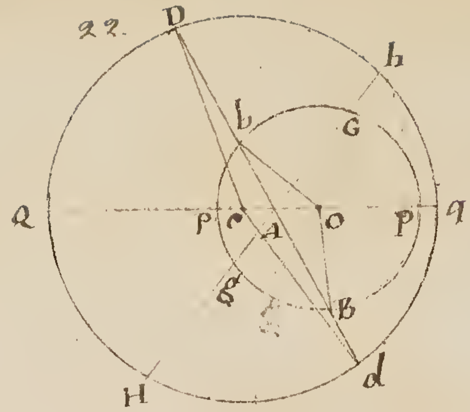
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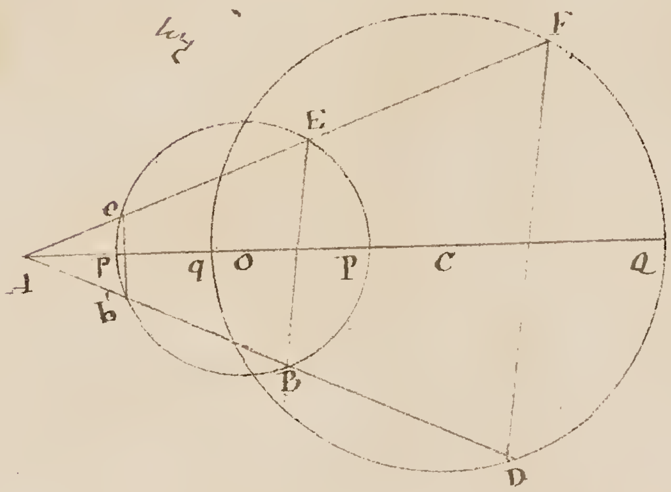
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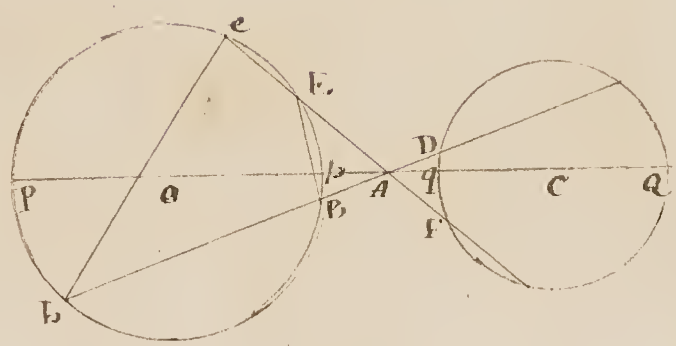
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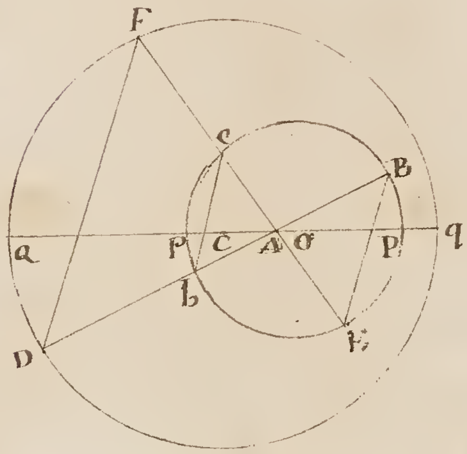
P
O
p
A
q
C
Q
B
F
b


F
e
B
Q
p
C
A
O
P
q
b
D
E

For let each of the right lines cut the Circle BCF again in B. E and join DF, BE, bc. Then since the right lines cut the Circles in unlike parts in D.C & F.e they will cut them in like parts in D.B & F. E and DF, BE will be parallel ( ). Therefore the external angle ABE is equal to the internal & opposite angle ADF ( ). But the Quadrilateral BEcb being inscribed in a Circle the angle ABE is equal to the external angle Acb or they are equal to each other because standing upon the same Arch. For the same reason is the angle AFD equal to the angle Abc therefore the triangles ADF, Acb are equiangular and their homologous sides are proportional ( ) viz. AD is to AF as Ac is to Ab and consequently the rectangle under AD. Ab is equal to the rectangle under AF. Ac.

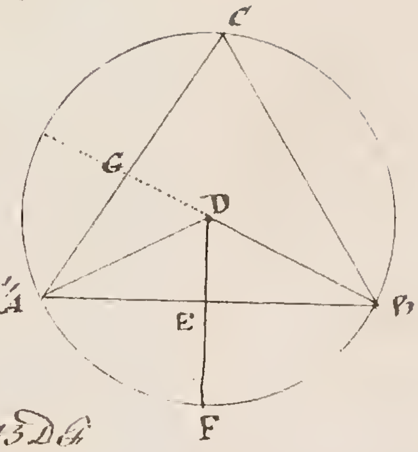
$\triangle$ [illegible] triangle
$\angle$ [illegible] angle
$\bigcirc$ [illegible] circumference
$\odot$ [illegible] circle
$\perp$ [illegible] perpendicular
e [illegible] Euclid

Trigonometry

Prop. A.

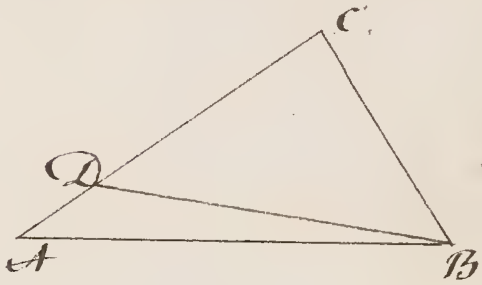
[illegible] ABC [illegible] AB [illegible] AC [illegible] ACB [illegible] ABC.

[illegible] ABC [illegible] D [illegible] DE ⊥ [illegible] AB [illegible] G, [illegible] AD, DB [illegible] DE [illegible] D [illegible] AB [illegible] AE = EB [illegible] AB [illegible] ADB [illegible] AG = [illegible] GB [illegible] ADG = [illegible] BDG [illegible] ADB [illegible] ADG, GDB [illegible] ADB [illegible] ACB [illegible] (e. 20. 3.) [illegible] ACB = ∠ ADG, GDB [illegible] AE [illegible] ADG [illegible] DA [illegible] AE [illegible] ACB; [illegible] DG [illegible] AC [illegible] CG, GA [illegible] CBA; [illegible] AB [illegible] AE [illegible] C [illegible] AC [illegible] AG [illegible] B; [illegible] i.e. AC [illegible] AB [illegible] AG [illegible] CBA [illegible] AE [illegible] ACB



Prop. B

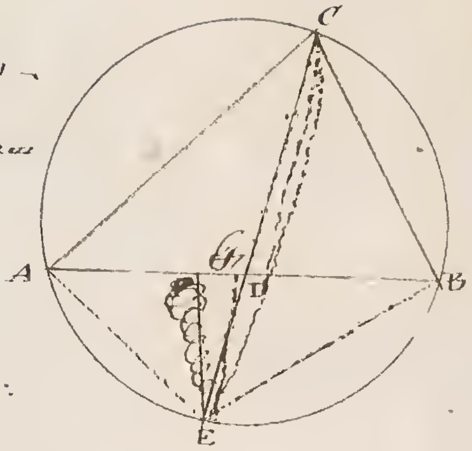
[illegible]
[illegible] AC [illegible] CB, [illegible] CD = CB; [illegible] DB, DBA [illegible] $\frac{1}{2}$ [illegible] CBA, CAB [illegible] if in any triangle you set off the less upon the greater side and join the point so found and the angle opposite the greater side the angle contained under that line and the base is $\frac{1}{2}$ the difference of the angles at the Base.



[illegible] CAB [illegible] CAB & CBA [illegible]
[illegible] C [illegible] AC [illegible] CD = CB [illegible] DB, DBA [illegible] $\frac{1}{2}$ [illegible] CAB, CBA [illegible] CBA [illegible] CAB
[illegible] CD, CB [illegible] CDB = CBD (e. 5. 1), ; CDB = ∠ DAB + DBA (e. 32. 1)
[illegible] CBD = DAB + DBA [illegible] DBA [illegible], CBA = DAB + 2DBA
[illegible] CBA − DBA = 2DBA [illegible] DBA = $\frac{1}{2}$ [illegible] CBA, CAB
Q. E. D

Prop. 1st

[illegible]



[illegible] ABC [illegible] AC+BC : AC − BC ::

S. ½ ∠A + ½ ∠B : S. ½ ∠B − ½ ∠A, [illegible] : CS. ½ ∠C :

S. ∠B − Co. ½ ∠C (S. Co ½ C − ∠A.

[illegible] ∠ACB [illegible] AB − D [illegible] − E

[illegible] AE [illegible] BE [illegible], EF [illegible] AB [illegible] AB − F

[illegible] ∠AEC = ∠B [illegible] ∠BEC = ∠A [illegible] ∠AEB = ∠A + ∠B, [illegible]

∠DEF = ½ [illegible] ∠B − ½ [illegible] ∠A ([illegible] ∠BEF = Co. ∠ABE =

Co. ½ ∠C, [illegible] ∠DEF = ∠BEF − ∠BED (Co ½ ∠C − ∠A) = ∠AEC − ∠AEF

(∠B − Co ½ ∠C) ; EF [illegible] ⊥ [illegible] BF, [illegible] BF.DF [illegible] ∠BEF

DEF ([illegible] EF) [illegible] BF : DF :: S. ∠BEF : S. ∠DEF. ; [illegible] ∠C [illegible]

[illegible] CD [illegible] AC : CB :: AD : DB [illegible] AC + CB : AC − CB ::

AB : AD − DB :: BF : DF :: S. ∠BEF : S. ∠DEF, [illegible] AC + BC :

AC − BC :: S ∠BEF : S. ∠DEF [illegible]

[illegible]

[illegible]

Prop. 2

[illegible]

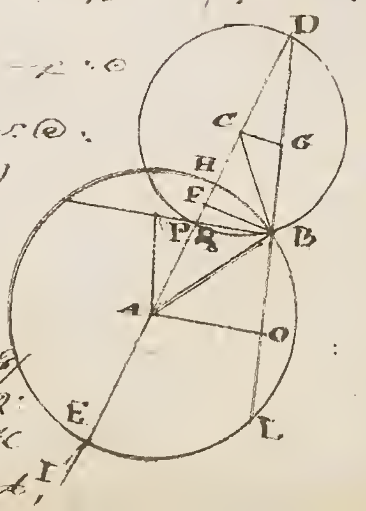
[illegible]

[illegible] ABC [illegible] AC, CB [illegible] AB [illegible] A [illegible] AB [illegible] EBH

[illegible] A, C − EH [illegible] AC [illegible] CD = CB / [illegible] ED = [illegible], ED − F

[illegible] EF = ½ [illegible] AF [illegible] AE [illegible] AB / [illegible] ACB :: [illegible] EFA ::

[illegible] ∠ACB / − AC [illegible] CD = CB [illegible] DB [illegible]



[illegible] AO, CG [illegible] ⊥ [illegible] DB, BF [illegible] AC, [illegible] AG = AC, [illegible]

[illegible] CB [illegible] CD [illegible] DBL / [illegible] BL [illegible] BO (3.E.3)

[illegible] BL [illegible] BG, [illegible] DL [illegible] OG [illegible] IC [illegible]

AC [illegible] IC : DL :: AC : OG [illegible] :: CD : DG [illegible] :: DB : DG.

[illegible] IC, DL [illegible] = ([illegible] ∠DB [illegible] (Cor. 36.E.3) [illegible] EDH /

[illegible] IC, DL : EDH (:: IC, DL : IC, DG [illegible] :: DL : DG

; DG [illegible] ∠ACB [illegible] CB [illegible] DL [illegible] IC, DL :

: EDH :: [illegible] ∠ACB [illegible] O / ; IC

[illegible] AC, DL [illegible] CB, ED [illegible] EF [illegible] ED [illegible] EF [illegible] EH [illegible] EA,

DH ... AP ... ACB :

EPA :: ... ACB /

Prop. 3.

... CD ... CB ...

... ED ... PD ... ED

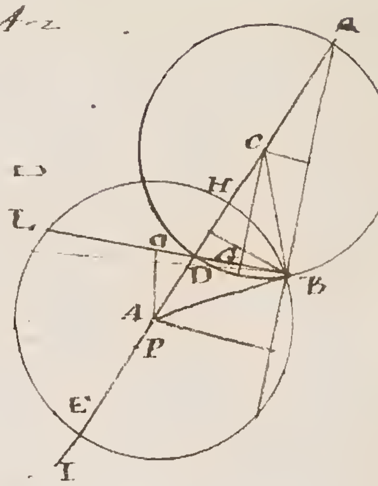
... PD ... EQ ... DC ... EQ ... PC viz PC ...

... PD ... EP ... PC ... CD ... CB / ... PA ...

... PC ... AC ... EH ... EA ... ED ... EP ...

DH ... PA / ... ACB ...

EPA ... vine ... ACB /



... BL ... BO ... BD ... BG ...

DL ... OG / ... DG ... verved vine

... ACB ... CB / ... ACB : ... EPA :: ...

... ACB / Prop 4th.

...

... (2) ...

... ex equo &c.

Prop. 5.

...

... (3)

... ex equo &c /

Prop. 6.

...

... (prop 4) ...

... (5) ...

... ex aequo ...

... viz)

... TU ...